

FUNCTION SPACES OF VARIABLE SMOOTHNESS AND INTEGRABILITY

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ABSTRACT. In this article we introduce Triebel–Lizorkin spaces with variable smoothness and integrability. Our new scale covers spaces with variable exponent as well as spaces of variable smoothness that have been studied in recent years. Vector-valued maximal inequalities do not work in the generality which we pursue, and an alternate approach is thus developed. Applying it, we give molecular and atomic decomposition results and show that our space is well-defined, i.e., independent of the choice of basis functions.

As in the classical case, a unified scale of spaces permits clearer results in cases where smoothness and integrability interact, such as Sobolev embedding and trace theorems. As an application of our decomposition we prove optimal trace theorems in the variable indices case.

1. INTRODUCTION

From a vast array of different function spaces a well ordered superstructure appeared in the 1960's and 70's based on two three-index spaces: the Besov space $B_{p,q}^\alpha$ and the Triebel–Lizorkin space $F_{p,q}^\alpha$. In recent years there has been a growing interest in generalizing classical spaces such as Lebesgue and Sobolev spaces to the case with either variable integrability (e.g., $W^{1,p(\cdot)}$) or variable smoothness (e.g., $W^{m(\cdot),2}$). These generalized spaces are obviously not covered by the superstructures with fixed indices.

It is well-known from the classical case that smoothness and integrability often interact, for instance, in trace and embedding theorems. However, there has so far been no attempt to treat spaces with variable integrability and smoothness in one scale. In this article we address this issue by introducing Triebel–Lizorkin spaces with variable indices, denoted $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

Spaces of variable integrability can be traced back to 1931 and W. Orlicz [41], but the modern development started with the paper [30] of Kováčik and Rákosník in 1991. A survey of the history of the field with a bibliography of more than a hundred titles published up to 2004 can be found in [17] by Diening, Hästö & Nekvinda; further surveys are due to Samko [49] and Mingione [42]. Apart from interesting theoretical considerations, the motivation to study such function spaces comes from applications to fluid dynamics, image processing, PDE and the calculus of variation.

The first concrete application arose from a model of electrorheological fluids in [45] (cf. [1, 2, 47, 48] for mathematical treatments of the model). To give the reader a feeling for the idea behind this application we mention that an electrorheological fluid is a so-called smart material in which the viscosity depends on the external electric field. This dependence is expressed through the variable exponent p ; specifically, the motion of the fluid is described by a Navier–Stokes-type equation where the Laplacian Δu is replaced by the $p(x)$ -Laplacian $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. By standard arguments, this means that the natural energy space of the problem is $W^{1,p(\cdot)}$, the Sobolev space of variable integrability. For further investigations of these differential equations see, e.g., [3, 18, 19].

2000 *Mathematics Subject Classification.* 46E35; 46E30, 42B15, 42B25.

Key words and phrases. Triebel–Lizorkin spaces, variable indices, non-standard growth, decomposition, molecule, atom, trace spaces.

* Supported in part by the Landesstiftung Baden-Württemberg.

† Supported in part by the Academy of Finland.

‡ Partially supported by the NSF grant DMS-0531337.

More recently, an application to image restoration was proposed by Chen, Levine & Rao [10, 40]. Their model combines isotropic and total variation smoothing. In essence, their model requires the minimization over u of the energy

$$\int_{\Omega} |\nabla u(x)|^{p(x)} + \lambda |u(x) - I(x)|^2 dx,$$

where I is given input. Recall that in the constant exponent case, the power $p \equiv 2$ corresponds to isotropic smoothing, whereas $p \equiv 1$ gives total variation smoothing. Hence the exponent varies between these two extremes in the variable exponent model. This variational problem has an Euler-Lagrange equation, and the solution can be found by solving a corresponding evolutionary PDE.

Partial differential equations have also been studied from a more abstract and general point of view in the variable exponent setting. In analogy to the classical case, we can approach boundary value problems through a suitable trace space, which, by definition, is a space consisting of restrictions of functions to the boundary. For the Sobolev space $W^{1,p(\cdot)}$, the trace space was first characterized by first two authors by an intrinsic norm, see [16]. In analogy with the classical case, this trace space can be formally denoted $F^{1-1/p(\cdot)}_{p(\cdot),p(\cdot)}$, so it is an example of a space with variable smoothness and integrability, albeit on with a very special relationship between the two exponents. Already somewhat earlier Almeida & Samko [4] and Gurka, Harjulehto & Nekvinda [26] had extended variable integrability Sobolev spaces to Bessel potential spaces $W^{\alpha,p(\cdot)}$ for constant but non-integer α .¹

Along a different line of study, Leopold [34, 35, 36, 37] and Leopold & Schrohe [38] studied pseudo-differential operators with symbols of the type $\langle \xi^{m(x)} \rangle$, and defined related function spaces of Besov-type with variable smoothness, formally $B^{m(\cdot)}_{p,p}$. In the case $p = 2$, this corresponds to the Sobolev space $H^{m(\cdot)} = W^{m(x),2}$. Function spaces of variable smoothness have recently been studied by Besov [5, 6, 7, 8]. He generalized Leopold's work by considering both Triebel–Lizorkin spaces $F^{s(\cdot)}_{p,q}$ and Besov spaces $B^{s(\cdot)}_{p,q}$ in \mathbb{R}^n . In a recent preprint, Schneider and Schwab [52] used $H^{m(\cdot)}(\mathbb{R})$ in the analysis of certain Black–Scholes equations. In this application the variable smoothness corresponds to the volatility of the market, which surely should change with time.

The purpose of the present paper is to define and study a generalized scale of Triebel–Lizorkin type spaces with variable smoothness, $\alpha(x)$, and variable primary and secondary indices of integrability, $p(x)$ and $q(x)$. By setting some of the indices to appropriate values we recover all previously mentioned spaces as special cases, except the Besov spaces (which, like in the classical case, form a separate scale).

Apart from the value added through unification, our new space allows treating traces and embeddings in a uniform and comprehensive manner, rather than doing them case by case. Some particular examples are:

- The trace space of $W^{k,p(\cdot)}$ is no longer a space of the same type. So, if we were interested in the trace space of the trace space, the theory of [16] no longer applies, and thus, a new theory is needed. In contrast to this, as we show in Section 7, the trace of a Triebel–Lizorkin space is again a Triebel–Lizorkin space (also in the variable indices case), hence, no such problem occurs.
- Our approach allows us to use the so-called “ r -trick” (cf. Lemma A.7) to study spaces with integrability in the range $(0, \infty]$, rather than in the range $[1, \infty]$.
- It is well-known that the constant exponent Triebel–Lizorkin space $F^0_{p,2}$ corresponds to the Hardy space H^p when $p \in (0, 1]$. Hardy spaces have thus far not

¹After the completion of this paper we learned that Xu [56, 57] has studied Besov and Triebel–Lizorkin spaces with variable p , but fixes q and α . The results in two subsections of Section 4 were proved independently in [57]. However, most of the advantages of unification do not occur with only p variable: for instance, trace spaces cannot be covered, and spaces of variable smoothness are not included. Therefore Xu's work does not essentially overlap with the results presented here.

been studied in the variable exponent case. Therefore, our formulation opens the door to this line of investigation.

When generalizing Triebel–Lizorkin spaces, we have several obstacles to overcome. The main difficulty is the absence of the vector-valued maximal function inequalities. It turns out that the inequalities are not only missing, rather, they do not even hold in the variable indices case (see Section 5). As a consequence of this, the Hörmander–Mikhlin multiplier theorem does not apply in the case of variable indices. Our solution is to work in closer connection with the actual structure of the space with what we call η -functions and to derive suitable estimates directly for these functions.

The structure of the article is as follows: we first briefly recapitulate some standard definitions and results in the next section. In Section 3 we state our main results: atomic and molecular decomposition of Triebel–Lizorkin spaces, a trace theorem, and a multiplier theorem. In Section 4 we show that our new scale is indeed a unification of previous spaces, in that it includes them all as special cases with appropriate choices of the indices. In Section 5 we formulate and prove an appropriate version of the multiplier theorem. In Section 6 we give the proofs of the main decompositions theorems, and in Section 7 we discuss the trace theorem. Finally, in Appendix A we derive several technical lemmas that were used in the other sections.

2. PRELIMINARIES

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B^n(x, r)$ the open ball in \mathbb{R}^n with center x and radius r . By B^n we denote the unit ball $B^n(0, 1)$. We use c as a generic constant, i.e., a constant whose values may change from appearance to appearance. The inequality $f \approx g$ means that $\frac{1}{c}g \leq f \leq cg$ for some suitably independent constant c . By χ_A we denote the characteristic function of the set A . If $a \in \mathbb{R}$, then we use the notation a_+ for the positive part of a , i.e., $a_+ = \max\{0, a\}$. By \mathbb{N} and \mathbb{N}_0 we denote the sets of positive and non-negative integers. For $x \in \mathbb{R}$ we denote by $[x]$ the largest integer less than or equal to x .

We denote the mean-value of the integrable function f , defined on a set A of finite, non-zero measure, by

$$\oint_A f(x) dx = \frac{1}{|A|} \int_A f(x) dx.$$

The Hardy–Littlewood maximal operator M is defined on $L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{r>0} \oint_{B^n(x,r)} |f(y)| dy.$$

By $\text{supp } f$ we denote the support of the function f , i.e., the closure of its zero set.

Spaces of variable integrability. By $\Omega \subset \mathbb{R}^n$ we always denote an open set. By a *variable exponent* we mean a measurable bounded function $p: \Omega \rightarrow (0, \infty)$ which is bounded away from zero. For $A \subset \Omega$ we denote $p_A^+ = \text{ess sup}_A p(x)$ and $p_A^- = \text{ess inf}_A p(x)$; we abbreviate $p^+ = p_\Omega^+$ and $p^- = p_\Omega^-$. We define *the modular* of a measurable function f to be

$$\varrho_{L^{p(\cdot)}(\Omega)}(f) = \int_\Omega |f(x)|^{p(x)} dx.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions $f: \Omega \rightarrow \mathbb{R}$ for which $\varrho_{L^{p(\cdot)}(\Omega)}(f) < \infty$. We define *the Luxemburg norm* on this space by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \varrho_{L^{p(\cdot)}(\Omega)}(f/\lambda) \leq 1 \},$$

which is the Minkowski functional of the absolutely convex set $\{f : \varrho_{L^{p(\cdot)}(\Omega)}(f) \leq 1\}$. In the case when $\Omega = \mathbb{R}^n$ we replace the $L^{p(\cdot)}(\mathbb{R}^n)$ in subscripts simply by $p(\cdot)$, e.g. $\|f\|_{p(\cdot)}$ denotes

$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the subspace of $L^{p(\cdot)}(\Omega)$ of functions f whose distributional gradient exists and satisfies $|\nabla f| \in L^{p(\cdot)}(\Omega)$. The norm

$$\|f\|_{W^{1,p(\cdot)}(\Omega)} = \|f\|_{L^{p(\cdot)}(\Omega)} + \|\nabla f\|_{L^{p(\cdot)}(\Omega)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

For fixed exponent spaces we of course have a very simple relationship between the norm and the modular. In the variable exponent case this is not so. However, we have nevertheless the following useful property: $\varrho_{p(\cdot)}(f) \leq 1$ if and only if $\|f\|_{p(\cdot)} \leq 1$. This and many other basic results were proven in [30].

Definition 2.1. Let $g \in C(\mathbb{R}^n)$. We say that g is *locally log-Hölder continuous*, abbreviated $g \in C_{\text{loc}}^{\log}(\mathbb{R}^n)$, if there exists $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$.

We say that g is *globally log-Hölder continuous*, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if it is locally log-Hölder continuous and there exists $g_{\infty} \in \mathbb{R}$ such that

$$|g(x) - g_{\infty}| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$.

Note that g is globally log-Hölder continuous if and only if

$$|g(x) - g(y)| \leq \frac{c}{|\log \frac{1}{2}q(x, y)|}$$

for all $x, y \in \overline{\mathbb{R}^n}$, where q denotes the spherical-chordal metric (the metric inherited from a projection to the Riemann sphere), hence the name, global log-Hölder continuity.

Building on [12] and [13] it is shown in [15, Theorem 3.6] that

$$M: L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n)$$

is bounded if $p \in C^{\log}(\mathbb{R}^n)$ and $1 < p^- \leq p^+ \leq \infty$. Global log-Hölder continuity is the best possible modulus of continuity to imply the boundedness of the maximal operator, see [12, 44]. However, if one moves beyond assumptions based on continuity moduli, it is possible to derive results also under weaker assumptions, see [14, 39, 43].

Partitions. Let \mathcal{D} be the collection of dyadic cubes in \mathbb{R}^n and denote by \mathcal{D}^+ the subcollection of those dyadic cubes with side-length at most 1. Let $\mathcal{D}_\nu = \{Q \in \mathcal{D} : \ell(Q) = 2^{-\nu}\}$. For a cube Q let $\ell(Q)$ denote the side length of Q and x_Q the “lower left corner”. For $c > 0$, we let cQ denote the cube with the same center and orientation as Q but with side length $c\ell(Q)$.

The set \mathcal{S} denotes the usual Schwartz space of rapidly decreasing complex-valued functions and \mathcal{S}' denotes the dual space of tempered distributions. We denote the Fourier transform of φ by $\hat{\varphi}$ or $\mathcal{F}\varphi$.

Definition 2.2. We say a pair (φ, Φ) is *admissible* if $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

- $\text{supp } \hat{\varphi} \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$ and $|\hat{\varphi}(\xi)| \geq c > 0$ when $\frac{3}{2} \leq |\xi| \leq \frac{5}{3}$,
- $\text{supp } \hat{\Phi} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$ and $|\hat{\Phi}(\xi)| \geq c > 0$ when $|\xi| \leq \frac{5}{3}$.

We set $\varphi_\nu(x) = 2^{\nu n} \varphi(2^\nu x)$ for $\nu \in \mathbb{N}$ and $\varphi_0(x) = \Phi(x)$. For $Q \in \mathcal{D}_\nu$ we set

$$\varphi_Q(x) = \begin{cases} |Q|^{1/2} \varphi_\nu(x - x_Q) & \text{if } \nu \geq 1, \\ |Q|^{1/2} \Phi(x - x_Q) & \text{if } \nu = 0. \end{cases}$$

We define ψ_ν and ψ_Q analogously.

Following [23], given an admissible pair (φ, Φ) we can select another admissible pair (ψ, Ψ) such that

$$\hat{\Phi}(\xi) \cdot \hat{\Psi}(\xi) + \sum_{\nu \geq 1} \hat{\Phi}(2^{-\nu}\xi) \cdot \hat{\Psi}(2^{-\nu}\xi) = 1 \quad \text{for all } \xi.$$

Here, $\tilde{\Phi}(x) = \overline{\Phi(-x)}$ and similarly for $\tilde{\varphi}$.

For each $f \in \mathcal{S}'(\mathbb{R}^n)$ we define the (inhomogeneous) φ -transform S_φ as the map taking f to the sequence $(S_\varphi f)_{Q \in \mathcal{D}^+}$ by setting $(S_\varphi f)_Q = \langle f, \varphi_Q \rangle$. Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $L^2(\mathbb{R}^n; \mathbb{C})$. For later purposes note that $(S_\varphi f)_Q = |Q|^{1/2} \tilde{\varphi}_\nu * f(2^{-\nu}k)$ for $l(Q) = 2^{-\nu} < 1$ and $(S_\varphi f)_Q = |Q|^{1/2} \tilde{\Phi} * f(2^{-\nu}k)$ for $l(Q) = 1$.

The inverse (inhomogeneous) φ -transform T_ψ is the map taking a sequence $s = \{s_Q\}_{l(Q) \leq 1}$ to $T_\psi s = \sum_{l(Q)=1} s_Q \Psi_Q + \sum_{l(Q)<1} s_Q \psi_Q$. We have the following identity for $f \in \mathcal{S}'(\mathbb{R}^n)$:

$$(2.3) \quad f = \sum_{Q \in \mathcal{D}_0} \langle f, \Phi_Q \rangle \Psi_Q + \sum_{\nu=1}^{\infty} \sum_{Q \in \mathcal{D}_\nu} \langle f, \varphi_Q \rangle \psi_Q.$$

Note that we consider all distributions in $\mathcal{S}'(\mathbb{R}^n)$ (rather than \mathcal{S}'/\mathcal{P} as in the homogeneous case), since $\hat{\Phi}(0) \neq 0$.

Using the admissible functions (φ, Φ) we can define the norms

$$\|f\|_{F_{p,q}^\alpha} = \left\| \left\| 2^{\nu\alpha} \varphi_\nu * f \right\|_{l_q} \right\|_{L^p} \quad \text{and} \quad \|f\|_{B_{p,q}^\alpha} = \left\| \left\| 2^{\nu\alpha} \varphi_\nu * f \right\|_{L^p} \right\|_{l_q},$$

for constants $p, q \in (0, \infty)$ and $\alpha \in \mathbb{R}$. The Triebel–Lizorkin space $F_{p,q}^\alpha$ and the Besov space $B_{p,q}^\alpha$ consists of distributions $f \in \mathcal{S}'$ for which $\|f\|_{F_{p,q}^\alpha} < \infty$ and $\|f\|_{B_{p,q}^\alpha} < \infty$, respectively. The classical theory of these spaces is presented for instance in the books of Triebel [53, 54, 55]. The discrete representation as sequence spaces through the φ -transform is due to Frazier and Jawerth [22, 23]. Recently, anisotropic and weighted versions of these spaces have been studied by many people, see, e.g., Bownik and Ho [9], Frazier and Roudenko [46, 24], Kühn, Leopold, Sickel and Skrzypczak [31], and the references therein. We now move on to generalizing these definitions to the variable index case.

3. STATEMENT OF THE MAIN RESULTS

In this section we introduce the main tool of this paper, a decomposition of the Triebel–Lizorkin space into molecules or atoms and state other important results. Section 4 contains further main results: there we show that previously studied spaces are indeed included in our scale. The proofs of the results from this section constitute much of the remainder of this article.

Throughout the paper we use the following

Standing Assumptions. We assume that p, q are positive functions on \mathbb{R}^n such that $\frac{1}{p}, \frac{1}{q} \in C^{\log}(\mathbb{R}^n)$. This implies, in particular, $0 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$. We also assume that $\alpha \in C_{\text{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\alpha \geq 0$ and that α has a limit at infinity.

One of the central classical tools that we are missing in the variable integrability setting is a general multiplier theorem of Mihlin–Hörmander type. We show in Section 5 that a general theorem does not hold, and instead prove the following result which is still sufficient to work with Triebel–Lizorkin spaces.

For a family of functions $f_\nu : \mathbb{R}^n \rightarrow \mathbb{R}$, $\nu \geq 0$, we define

$$\|f_\nu(x)\|_{l_v^{q(x)}} = \left(\sum_{\nu \geq 0} |f_\nu(x)|^{q(x)} \right)^{\frac{1}{q(x)}}.$$

Note that this is just an ordinary discrete Lebesgue space, since $q(x)$ does not depend on v . The mapping $x \mapsto \|f_v(x)\|_{\ell_v^{q(x)}}$ is a function of x and can be measured in $L^{p(\cdot)}$. We write $L_x^{p(\cdot)}$ to indicate that the integration variable is x . We define

$$(3.1) \quad \eta_m(x) = (1 + |x|)^{-m} \quad \text{and} \quad \eta_{v,m}(x) = 2^{nv} \eta_m(2^v x).$$

Theorem 3.2. *Let $p, q \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $1 < q^- \leq q^+ < \infty$. Then the inequality*

$$\left\| \left\| \eta_{v,m} * f_v \right\|_{\ell_v^{q(x)}} \right\|_{L_x^{p(\cdot)}} \leq c \left\| \left\| f_v \right\|_{\ell_v^{q(x)}} \right\|_{L_x^{p(\cdot)}}$$

holds for every sequence $\{f_v\}_{v \in \mathbb{N}_0}$ of L_{loc}^1 -functions and constant $m > n$.

Definition 3.3. Let $\varphi_v, v \in \mathbb{N}_0$, be as in Definition 2.2. The Triebel–Lizorkin space $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is defined to be the space of all distributions $f \in \mathcal{S}'$ with $\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} < \infty$, where

$$\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} := \left\| \left\| 2^{v\alpha(x)} \varphi_v * f(x) \right\|_{\ell_v^{q(x)}} \right\|_{L_x^{p(\cdot)}}.$$

In the case of $p = q$ we use the notation $F_{p(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) := F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Note that, *a priori*, the function space depends on the choice of admissible functions (φ, Φ) . One of the main purposes of this paper is to show that, up to equivalence of norms, every pair of admissible functions produces the same space.

In the classical case it has proved very useful to express the Triebel–Lizorkin norm in terms of two sums, rather than a sum and an integral, thus, giving rise to discrete Triebel–Lizorkin spaces $f_{p,q}^\alpha$. Intuitively, this is achieved by viewing the function as a constant on dyadic cubes. The size of the appropriate dyadic cube varies according to the level of smoothness.

We next present a formulation of the Triebel–Lizorkin norm which is similar in spirit.

For a sequence of real numbers $\{s_Q\}_Q$ we define

$$\|\{s_Q\}_Q\|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} := \left\| \left\| 2^{v\alpha(x)} \sum_{Q \in \mathcal{D}_v} |s_Q| |Q|^{-\frac{1}{2}} \chi_Q \right\|_{\ell_v^{q(x)}} \right\|_{L_x^{p(\cdot)}}.$$

The space $f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ consists of all those sequences $\{s_Q\}_Q$ for which this norm is finite. We are ready to state our first decomposition result, which says that $S_\varphi : F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} \hookrightarrow f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ is a bounded operator.

Theorem 3.4. *If p, q and α are as in the Standing Assumptions, then*

$$\|S_\varphi f\|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \leq c \|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

If we have a sequence $\{s_Q\}_Q$, then we can easily construct a candidate Triebel–Lizorkin function by taking the weighted sum with certain basis functions, $\sum s_Q m_Q$. Obviously, certain restrictions are necessary on the functions m_Q in order for this to work. We therefore make the following definitions:

Definition 3.5. Let $v \in \mathbb{N}_0$, $Q \in \mathcal{D}_v$ and $k \in \mathbb{Z}$, $l \in \mathbb{N}_0$ and $M \geq n$. A function m_Q is said to be a (k, l, M) -smooth molecule for Q if it satisfies the following conditions for some $m > M$:

- (M1) if $v > 0$, then $\int_{\mathbb{R}^n} x^\gamma m_Q(x) dx = 0$ for all $|\gamma| \leq k$; and
- (M2) $|D^\gamma m_Q(x)| \leq 2^{|\gamma|v} |Q|^{1/2} \eta_{v,m}(x + x_Q)$ for all multi-indices $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq l$.

The conditions (M1) and (M2) are called the *moment* and *decay conditions*, respectively.

Note that (M1) is vacuously true if $k < 0$. When $M = n$, this definition is a special case of the definition given in [23] for molecules. The difference is that we consider only k and l integers, and l non-negative. In this case two of the four conditions given in [23] are vacuous.

Definition 3.6. Let $K, L: \mathbb{R}^n \rightarrow \mathbb{R}$ and $M > n$. The family $\{m_Q\}_Q$ is said to be a *family of (K, L, M) -smooth molecules* if m_Q is $(\lfloor K_Q^-, \lfloor L_Q^-, M)$ -smooth for every $Q \in \mathcal{D}^+$.

Definition 3.7. We say that $\{m_Q\}_Q$ is a *family of smooth molecules for $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$* if it is a family of $(N + \varepsilon, \alpha + 1 + \varepsilon, M)$ -smooth molecules, where

$$N(x) := \frac{n}{\min\{1, p(x), q(x)\}} - n - \alpha(x),$$

for some constant $\varepsilon > 0$, and M is a sufficiently large constant.

The number M needs to be chosen sufficiently large, for instance

$$2 \frac{n + c_{\log}(\alpha)}{\min\{1, p^-, q^-\}}$$

will do, where $c_{\log}(\alpha)$ denotes the log-Hölder continuity constant of α . Since M can be fixed depending on the parameters we will usually omit it from our notation of molecules.

Note that the functions φ_Q are smooth molecules for arbitrary indices. Also note that compared to the classical case we assume the existence of 1 more derivative (rounded down) for smooth molecules for $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$. We need the assumption for technical reasons (cf. Lemma 6.2). However, we think the additional assumptions are inconsequential; for instance the trace result (Theorem 3.13), and indeed any result based on atomic decomposition, can still be proven in an optimal form.

Theorem 3.8. *Let the functions p, q , and α be as in the Standing Assumptions. Suppose that $\{m_Q\}_Q$ is a family of smooth molecules for $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ and that $\{s_Q\}_Q \in f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$. Then*

$$\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \leq c \|\{s_Q\}_Q\|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}, \quad \text{where } f = \sum_{\nu \geq 0} \sum_{Q \in \mathcal{D}_\nu} s_Q m_Q.$$

Theorems 3.4 and 3.8 yield an isomorphism between $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ and a subspace of $f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ via the S_φ transform:

Corollary 3.9. *If the functions p, q , and α are as in the Standing Assumptions, then*

$$\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \|S_\varphi f\|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}$$

for every $f \in F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

With these tools we can prove that the space $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is well-defined.

Theorem 3.10. *The space $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$ is well-defined, i.e., the definition does not depend on the choice of the functions φ and Φ satisfying the conditions of Definition 2.2, up to the equivalence of norms.*

Proof. Let $\tilde{\varphi}_\nu$ and φ_ν be different basis functions as in Definition 2.2. Let $\|\cdot\|_{\tilde{\varphi}}$ and $\|\cdot\|_\varphi$ denote the corresponding norms of $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$. By symmetry, it suffices to prove $\|f\|_{\tilde{\varphi}} \leq c \|f\|_\varphi$ for all $f \in \mathcal{S}'$. Let $\|f\|_\varphi < \infty$. Then by (2.3) and Theorem 3.4 we have $f = \sum_{Q \in \mathcal{D}^+} (S_\varphi f)_Q \psi_Q$ and $\|S_\varphi f\|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \leq c \|f\|_\varphi$. Since $\{\psi_Q\}_Q$ is a family of smooth molecules, $\|f\|_{\tilde{\varphi}} \leq c \|S_\varphi f\|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}$ by Theorem 3.8, which completes the proof. \square

It is often convenient to work with compactly supported basis functions. Thus, we say that the molecule a_Q concentrated on Q is an *atom* if it satisfies $\text{supp } a_Q \subset 3Q$. The downside of atoms is that we need to choose a new set of them for each function f that we

represent. Note that this coincides with the definition of atoms in [23] in the case when p , q and α are constants.

For atomic decomposition we have the following result.

Theorem 3.11. *Let the functions p , q , and α be as in the Standing Assumptions and let $f \in F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$. Then there exists a family of smooth atoms $\{a_Q\}_Q$ and a sequence of coefficients $\{t_Q\}_Q$ such that*

$$f = \sum_{Q \in \mathcal{D}^+} t_Q a_Q \text{ in } \mathcal{S}' \quad \text{and} \quad \|\{t_Q\}_Q\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

Moreover, the atoms can be chosen to satisfy conditions (M1) and (M2) in Definition 3.5 for arbitrarily high, given order.

If the maximal operator is bounded and $1 < p^- \leq p^+ < \infty$, then it follows easily that $C_0^\infty(\mathbb{R}^n)$ (the space of smooth functions with compact support) is dense in $W^{1, p(\cdot)}(\mathbb{R}^n)$, since it is then possible to use convolution. However, density can be achieved also under more general circumstances, see [21, 29, 58]. Our standing assumptions are strong enough to give us density directly:

Corollary 3.12. *Let the functions p , q , and α be as in the Standing Assumptions. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.*

Another consequence of our atomic decomposition is the analogue of the standard trace theorem. Since its proof is much more involved, we present it in Section 7. Note that the assumption $\alpha - \frac{1}{p} - (n-1)\left(\frac{1}{p} - 1\right)_+ > 0$ is optimal also in the constant smoothness and integrability case, cf. [22, Section 5]

Theorem 3.13. *Let the functions p , q , and α be as in the Standing Assumptions. If*

$$\alpha - \frac{1}{p} - (n-1)\left(\frac{1}{p} - 1\right)_+ > 0, \quad \text{then} \quad \text{tr } F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n) = F_{p(\cdot)}^{\alpha(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1}).$$

4. SPECIAL CASES

In this section we show how the Triebel–Lizorkin scale $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ includes as special cases previously studied spaces with variable differentiability or integrability.

Lebesgue spaces. We begin with the variable exponent Lebesgue spaces from Section 2, which were originally introduced by Orlicz in [41]. We show that $F_{p(\cdot), 2}^0 \cong L^{p(\cdot)}$ under suitable assumptions on p . We use an extrapolation result for $L^{p(\cdot)}$. Recall, that a weight ω is in the Muckenhoupt class A_1 if $M\omega \leq K\omega$ for some such $K > 0$. The smallest K is the A_1 constant of ω .

Lemma 4.1 (Theorem 1.3, [11]). *Let $p \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and let \mathcal{G} denote a family of tuples (f, g) of measurable functions on \mathbb{R}^n . Suppose that there exists a constant $r_0 \in (0, p^-)$ so that*

$$\left(\int_{\mathbb{R}^n} |f(x)|^{r_0} \omega(x) dx \right)^{\frac{1}{r_0}} \leq c_0 \left(\int_{\mathbb{R}^n} |g(x)|^{r_0} \omega(x) dx \right)^{\frac{1}{r_0}}$$

for all $(f, g) \in \mathcal{G}$ and every weight $\omega \in A_1$, where c_0 is independent of f and g and depends on ω only via its A_1 -constant. Then

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c_1 \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for all $(f, g) \in \mathcal{G}$ with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$.

Theorem 4.2. *Let $p \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$. Then $L^{p(\cdot)}(\mathbb{R}^n) \cong F_{p(\cdot), 2}^0(\mathbb{R}^n)$. In particular,*

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \|\|\varphi_\nu * f\|_{L^2_\nu}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

Proof. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ (see [30]) and also in $F_{p(\cdot), 2}^0(\mathbb{R}^n)$ by Corollary 3.12, it suffices to prove the claim for all $f \in C_0^\infty(\mathbb{R}^n)$. Fix $r \in (1, p^-)$. Then

$$\|\|\varphi_\nu * f\|_{L^2_\nu}\|_{L^{p(\cdot)}(\mathbb{R}^n; \omega)} \approx \|f\|_{L^{r_0}(\mathbb{R}^n; \omega)},$$

for all $\omega \in A_1$ by [32, Theorem 1], where the constant depends only on the A_1 -constant of the weight ω , so the assumptions of Lemma 4.1 are satisfied. Applying the lemma with \mathcal{G} equal to either

$$\{(\|\varphi_\nu * f\|_{L^2_\nu}, f) : f \in C_0^\infty(\Omega)\} \quad \text{or} \quad \{(f, \|\varphi_\nu * f\|_{L^2_\nu}) : f \in C_0^\infty(\Omega)\}$$

completes the proof. \square

Theorem 4.2 generalizes the equivalence of $L^p(\mathbb{R}^n) \cong F_{p, q}^0$ for constant $p \in (1, \infty)$ to the setting of variable exponent Lebesgue spaces. If $p \in (0, 1]$, then the spaces $L^p(\mathbb{R}^n)$ have to be replaced by the Hardy spaces $h^p(\mathbb{R}^n)$. This suggests the following definition:

Definition 4.3. Let $p \in C^{\log}(\mathbb{R}^n)$ with $0 < p^- \leq p^+ < \infty$. Then we define the *variable exponent Hardy space* $h^{p(\cdot)}(\mathbb{R}^n)$ by $h^{p(\cdot)}(\mathbb{R}^n) := F_{p(\cdot), 2}^0(\mathbb{R}^n)$.

The investigation of this space is left for future research.

Sobolev and Bessel spaces. We move on to Bessel potential spaces with variable integrability, which have been independently introduced by Almeida & Samko [4] and Gurka, Harjulehto & Nekvinda [26]. This scale includes also the variable exponent Sobolev spaces $W^{k, p(\cdot)}$.

In the following let \mathcal{B}^σ denote the Bessel potential operator $\mathcal{B}^\sigma = \mathcal{F}^{-1}(1 + |\xi|^2)^{-\frac{\sigma}{2}}\mathcal{F}$ for $\sigma \in \mathbb{R}$. Then the *variable exponent Bessel potential space* is defined by

$$\mathcal{L}^{\alpha, p(\cdot)}(\mathbb{R}^n) := \mathcal{B}^\alpha(L^{p(\cdot)}(\mathbb{R}^n)) = \{\mathcal{B}^\alpha g : g \in L^{p(\cdot)}(\mathbb{R}^n)\}$$

equipped with the norm $\|g\|_{\mathcal{L}^{\alpha, p(\cdot)}} := \|\mathcal{B}^{-\alpha} g\|_{p(\cdot)}$. It was shown independently in [4, Corollary 6.2] and [26, Theorem 3.1] that $\mathcal{L}^{k, p(\cdot)}(\mathbb{R}^n) \cong W^{k, p(\cdot)}(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$ when $p \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$.

We will show that $\mathcal{L}^{\alpha, p(\cdot)}(\mathbb{R}^n) \cong F_{p(\cdot), 2}^\alpha(\mathbb{R}^n)$ under suitable assumptions on p for $\alpha \geq 0$ and that $\mathcal{L}^{k, p(\cdot)}(\mathbb{R}^n) \cong W^{k, p(\cdot)}(\mathbb{R}^n) \cong F_{p(\cdot), 2}^k(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$. It is clear by the definition of $\mathcal{L}^{\alpha, p(\cdot)}(\mathbb{R}^n)$ that \mathcal{B}^σ with $\sigma \geq 0$ is an isomorphism between $\mathcal{L}^{\alpha, p(\cdot)}(\mathbb{R}^n)$ and $\mathcal{L}^{\alpha+\sigma, p(\cdot)}(\mathbb{R}^n)$, i.e., it has a lifting property. Therefore, in view of Theorem 4.2 and $\mathcal{L}^{0, p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n) \cong F_{p(\cdot), 2}^0(\mathbb{R}^n)$, we will complete the circle by proving a lifting property for the scale $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\mathbb{R}^n)$.

Lemma 4.4 (Lifting property). *Let p , q , and α be as in the Standing Assumptions and $\sigma \geq 0$. Then the Bessel potential operator \mathcal{B}^σ is an isomorphism between $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ and $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)+\sigma}$.*

Proof. Let $f \in F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$. We know that $\{\varphi_Q\}$ is a family of smooth molecules, thus, by Theorem 3.4

$$\|\{s_Q\}_Q\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}},$$

where $f = \sum_{Q \in \mathcal{D}^+} s_Q \varphi_Q$. Therefore,

$$\mathcal{B}^\sigma f = \sum_{Q \in \mathcal{D}^+} s_Q \mathcal{B}^\sigma \varphi_Q = \sum_{Q \in \mathcal{D}^+} \underbrace{2^{-\nu\sigma} s_Q}_{=: s'_Q} \underbrace{2^{\nu\sigma} \mathcal{B}^\sigma \varphi_Q}_{=: \varphi'_Q}.$$

Let us check that $\{K\varphi'_Q\}_Q$ is a family of smooth molecules of an arbitrary order for a suitable constant K . Let $Q \in \mathcal{D}^+$. Without loss of generality we may assume that $x_Q = 0$. Then

$$\widehat{\varphi'_Q}(\xi) = \frac{2^{\nu\sigma}\widehat{\varphi_Q}(\xi)}{(1+|\xi|)^\sigma} = \frac{2^{\nu\sigma}|Q|^{1/2}\widehat{\varphi}(2^{-\nu}\xi)}{(1+|\xi|)^\sigma}.$$

Since $\widehat{\varphi}$ has support in the annulus $B^n(0, 2) \setminus B^n(0, 1/2)$, it is clear that $\widehat{\varphi'_Q} \equiv 0$ in a neighborhood of the origin when $l(Q) < 1$, so the family satisfies the moment condition in Definition 3.5 for an arbitrarily high order.

Next we consider the decay condition for molecules. Let $\mu \in \mathbb{N}_0^n$ be a multi-index with $|\mu| = m$. We estimate

$$\begin{aligned} |D_\xi^\mu \widehat{\varphi'_Q}(\xi)| &\leq 2^{\nu\sigma}|Q|^{1/2} \left| D_\xi^\mu \left[\frac{\widehat{\varphi}(2^{-\nu}\xi)}{(1+|\xi|)^\sigma} \right] \right| \\ &= |Q|^{1/2} 2^{-\nu m} \left| D_\xi^\mu \left[\frac{\widehat{\varphi}(\xi)}{(2^{-\nu} + |\xi|)^\sigma} \right] \right| \\ &\leq c |Q|^{1/2} 2^{-\nu m} |D_\xi^\mu [\widehat{\varphi}(\xi)|\xi|^{-\sigma}]|, \end{aligned}$$

where $\xi = 2^{-\nu}\xi$ and we used that the support of $\widehat{\varphi}$ lies in the annulus $B^n(0, 2) \setminus B^n(0, 1/2)$ for the last estimate. Define

$$K_m = \sup_{|\mu|=m, \xi \in \mathbb{R}^n} 2^{-\nu m} |D_\xi^\mu [\widehat{\varphi}(\xi)|\xi|^{-\sigma}]|.$$

Since $\sigma \geq 0$ and $\widehat{\varphi}$ vanishes in a neighborhood of the origin, we conclude that $K_m < \infty$ for every m . From the estimate

$$|x^\mu \psi(x)| = c \left| \int_{\mathbb{R}^n} (-1)^m D_\xi^\mu \widehat{\psi}(\xi) e^{ix \cdot \xi} d\xi \right| \leq c |\text{supp } \widehat{\psi}| \sup_\xi |D_\xi^\mu \widehat{\psi}(\xi)|,$$

we conclude that

$$|x^m| |\varphi'_Q(x)| \leq c 2^{\nu m} |Q|^{1/2} 2^{-\nu m} K_m \quad \text{and} \quad |\varphi'_Q(x)| \leq c 2^{\nu m} |Q|^{1/2} K_0.$$

Multiplying the former of the two inequalities by $2^{\nu m}$ and adding it to the latter gives

$$(1 + 2^{\nu m} |x|^m) |\varphi'_Q(x)| \leq c 2^{\nu m} |Q|^{1/2} (K_0 + K_m).$$

Finally, this implies that

$$|\varphi'_Q(x)| \leq c \frac{2^{\nu m}}{(1 + 2^{\nu m} |x|^m)^m} |Q|^{1/2} (K_0 + K_m) = |Q|^{1/2} (K_0 + K_m) \eta_{\nu, m}(x),$$

from which we conclude that the family $\{K\varphi'_Q\}_Q$ satisfy the decay condition when $K \leq (|Q|^{1/2} (K_0 + K_m))^{-1}$. A similar argument yields the decay condition for $D_x^\mu \varphi'_Q$.

Since $\{K\varphi'_Q\}_Q$ is a family of smooth molecules for $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)+\sigma}$, we can apply Theorem 3.8 to conclude that

$$\|\mathcal{B}^\sigma f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)+\sigma}} \leq c \|\{s'_Q/K\}_Q\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)+\sigma}} \leq c \|\{s_Q\}_Q\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}.$$

The reverse inequality is handled similarly. \square

Theorem 4.5. *Let $p \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $\alpha \in [0, \infty)$. Then $F_{p(\cdot), 2}^\alpha(\mathbb{R}^n) \cong \mathcal{L}^{\alpha, p(\cdot)}(\mathbb{R}^n)$. If $k \in \mathbb{N}_0$, then $F_{p(\cdot), 2}^k(\mathbb{R}^n) \cong W^{k, p(\cdot)}(\mathbb{R}^n)$.*

Proof. Suppose that $f \in F_{p(\cdot), 2}^\alpha(\mathbb{R}^n)$. By Lemma 4.4, $\mathcal{B}^{-\alpha} f \in F_{p(\cdot), 2}^0(\mathbb{R}^n)$, so we conclude by Theorem 4.2 that $\mathcal{B}^{-\alpha} f \in L^{p(\cdot)}(\mathbb{R}^n) = \mathcal{L}^{0, p(\cdot)}(\mathbb{R}^n)$. Then it follows by the definition of the Bessel space that $f = \mathcal{B}^\alpha [\mathcal{B}^{-\alpha} f] \in \mathcal{L}^{\alpha, p(\cdot)}(\mathbb{R}^n)$. The reverse inclusion follows by reversing these steps.

The claim regarding the Sobolev spaces follows from this and the equivalence $\mathcal{L}^{k, p(\cdot)}(\mathbb{R}^n) \cong W^{k, p(\cdot)}(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$ (see [4, Corollary 6.2] or [26, Theorem 3.1]). \square

Spaces of variable smoothness. Finally, we come to spaces of variable smoothness as introduced by Besov [5], following Leopold [33]. Let $p, q \in (1, \infty)$ and let $\alpha \in C_{\text{loc}}^{\log}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\alpha \geq 0$. Then Besov defines the following spaces of variable smoothness

$$F_{p,q}^{\alpha(\cdot), \text{Besov}}(\mathbb{R}^n) := \left\{ f \in L_{\text{loc}}^p(\mathbb{R}^n) : \|f\|_{F_{p,q}^{\alpha(\cdot), \text{Besov}}} < \infty \right\},$$

$$\|f\|_{F_{p,q}^{\alpha(\cdot), \text{Besov}}} := \left\| \left\| 2^{\nu\alpha(x)} \int_{|h| \leq 1} |\Delta^M(2^{-k}h, f)(x)| dh \right\|_{L_x^p} \right\|_{L_y^q} + \|f\|_{L_x^p},$$

where

$$\Delta^M(y, f)(x) := \sum_{k=0}^M (-1)^{M-k} \binom{M}{k} f(x + ky).$$

In [7, Theorem 3.2] Besov proved that $F_{p,q}^{\alpha(\cdot), \text{Besov}}(\mathbb{R}^n)$ can be renormed by

$$\left\| \left\| 2^{\nu\alpha(x)} \varphi_\nu * f(x) \right\|_{L_x^p} \right\|_{L_y^q} \approx \|f\|_{F_{p,q}^{\alpha(\cdot), \text{Besov}}},$$

which agrees with our definition of the norm of $F_{p,q}^{\alpha(\cdot)}$, since p and q are constants. This immediately implies the following result:

Theorem 4.6. *Let $p, q \in (1, \infty)$, $\alpha \in C_{\text{loc}}^{\log} \cap L^\infty$ and $\alpha \geq 0$. Then $\|f\|_{F_{p,q}^{\alpha(\cdot), \text{Besov}}(\mathbb{R}^n)} \approx \|f\|_{F_{p,q}^{\alpha(\cdot)}(\mathbb{R}^n)}$.*

In his works, Besov also studied Besov spaces of variable differentiability. For $p, q \in (1, \infty)$ and $\alpha \in C_{\text{loc}}^{\log} \cap L^\infty$ with $\alpha \geq 0$, he defines

$$B_{p,q}^{\alpha(\cdot), \text{Besov}}(\mathbb{R}^n) := \left\{ f \in L_{\text{loc}}^p(\mathbb{R}^n) : \|f\|_{B_{p,q}^{\alpha(\cdot), \text{Besov}}(\mathbb{R}^n)} < \infty \right\},$$

$$\|f\|_{B_{p,q}^{\alpha(\cdot), \text{Besov}}(\mathbb{R}^n)} := \left\| \left\| \sup_{|h| \leq 1} |\Delta^M(2^{-\nu\alpha(x)}h, f)(x)| \right\|_{L_x^p} \right\|_{L_y^q} + \|f\|_{L_x^p}.$$

Remark 4.7. In fact, Besov gives a slightly more general definition than this for both the Triebel–Lizorkin and the Besov spaces. He replaces $2^{\nu\alpha(x)}$ by a sequence of functions $\beta_\nu(x)$. The functions $\beta_\nu(x)$ are then assumed to satisfy some regularity assumptions with respect to ν and x , which are very closely related to the local log–Hölder continuity of α . Indeed, if $\beta_\nu(x) = 2^{\nu\alpha(x)}$, then his conditions on β_ν are precisely that $\alpha \in C_{\text{loc}}^{\log} \cap L^\infty$.

In the classical case the scale of Triebel–Lizorkin spaces and the scale of Besov spaces agree if $p = q$. Besov showed in [8] that this is also the case for his new scales of Triebel–Lizorkin and Besov spaces, i.e., $F_{p,p}^{\alpha(\cdot), \text{Besov}}(\mathbb{R}^n) = B_{p,p}^{\alpha(\cdot), \text{Besov}}(\mathbb{R}^n)$ for $p \in (1, \infty)$, $\alpha \in C_{\text{loc}}^{\log} \cap L^\infty$, and $\alpha \geq 0$. This enables us to point out a connection to another family of spaces. By means of the symbols of pseudodifferential operators, Leopold [33] introduced Besov spaces with variable differentiability $B_{p,p}^{\alpha(\cdot), \text{Leopold}}(\mathbb{R}^n)$. He further showed that if $0 < \alpha^- \leq \alpha^+ < \infty$ and $\alpha \in C^\infty(\mathbb{R}^n)$, then the spaces $B_{p,p}^{\alpha(\cdot), \text{Leopold}}(\mathbb{R}^n)$ can be characterized by means of finite differences. This characterization agrees with the one that later Besov [6] used in the definition of the spaces $B_{p,p}^{\alpha(\cdot), \text{Besov}}(\mathbb{R}^n)$. In particular, we have $B_{p,p}^{\alpha(\cdot), \text{Leopold}}(\mathbb{R}^n) = B_{p,p}^{\alpha(\cdot), \text{Besov}}(\mathbb{R}^n) = F_{p,p}^{\alpha(\cdot)}(\mathbb{R}^n)$ for such α .

Other spaces. It should be mentioned that there have recently also been some extensions of variable integrability spaces in other directions, not covered by the Triebel–Lizorkin scale that we introduce here. For instance, Harjulehto & Hästö [27] modified the Lebesgue space scale on the upper end to account for the fact that $W^{1,n}$ does not map to L^∞ under the Sobolev embedding. Similarly, in the image restoration model by Chen, Levine and Rao mentioned above, one has the problem that the exponent p takes values in the closed interval $[1, 2]$, including the lower bound, so that one is not working with reflexive spaces. It is well-known that the space BV of functions of bounded variation is often a better

alternative than $W^{1,1}$ when studying differential equations. Consequently, it was necessary to modify the scale $W^{1,p(\cdot)}$ so that the lower end corresponded to BV . This was done by Harjulehto, Hästö & Latvala in [28]. Schneider [50, 51] has also investigated spaces of variable smoothness, but these spaces are not included in the scale of Leopold and Besov. Most recently, Diening, Harjulehto, Hästö, Mizuta & Shimomura [15] have studied Sobolev embeddings when $p \rightarrow 1$ using Lebesgue spaces with an $L \log L$ -character on the lower end in place of L^1 .

5. MULTIPLIER THEOREMS

Cruz-Uribe, Fiorenza, Martell and Pérez [11, Corollary 2.1] proved a very general extrapolation theorem, which implies among other things the following vector-valued maximal inequality, for variable p but constant q :

Lemma 5.1. *Let $p \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leq p^+ < \infty$ and $1 < q < \infty$. Then*

$$\| \|Mf_i\|_{l^q} \|_{p(\cdot)} \leq C \| \|f_i\|_{l^q} \|_{p(\cdot)}.$$

It would be very nice to generalize this estimate to the variable q case. In particular, this would allow us to use classical machinery to deal with Triebel–Lizorkin spaces. Unfortunately, it turns out that it is not possible: if q is not constant, then the inequality

$$\| \|Mf_i\|_{l^q(\cdot)} \|_{L^{p(\cdot)}_x} \leq C \| \|f_i\|_{l^q(\cdot)} \|_{L^{p(\cdot)}_x}$$

does not hold, even if p is constant or $p(\cdot) = q(\cdot)$. For a concrete counter-example consider q with $q|_{\Omega_j} = q_j$, $j = 0, 1$, and $q_0 \neq q_1$ and a constant p . Set $f_k := a_k \chi_{\Omega_0}$. Then $Mf_k|_{\Omega_1} \geq c a_k \chi_{\Omega_1}$. This shows that $l^{q_0} \hookrightarrow l^{q_1}$. The opposite embedding follows in the same way, hence, we would conclude that $l^{q_0} \cong l^{q_1}$, which is of course false.

In lieu of a vector-valued maximal inequality, we prove in this section estimates which take into account that there is a clear stratification in the Triebel–Lizorkin space, namely, a given magnitude of cube size is used in exactly one term in the sum. Recall that $\eta_m(x) = (1 + |x|)^{-m}$ and $\eta_{\nu,m}(x) = 2^{n\nu} \eta_m(2^\nu x)$. For a measurable set Q and an integrable function g we denote

$$M_Q g := \int_Q |g(x)| dx.$$

Lemma 5.2. *For every $m > n$ there exists $c = c(m, n) > 0$ such that*

$$\eta_{\nu,m} * |g|(x) \leq c \sum_{j \geq 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) M_Q g$$

for all $\nu \geq 0$, $g \in L^1_{\text{loc}}$, and $x \in \mathbb{R}^n$.

Proof. Fix $\nu \geq 0$, $g \in L^1_{\text{loc}}$, and $x, y \in \mathbb{R}^n$. If $|x - y| \leq 2^{-\nu}$, then we choose $Q \in \mathcal{D}_\nu$ which contains x and y . If $|x - y| > 2^{-\nu}$, then we choose $j \in \mathbb{N}_0$ such that $2^{\nu-j} \leq |x - y| \leq 2^{\nu-j+1}$ and let $Q \in \mathcal{D}_{\nu-j}$ be the cube containing y . Note that $x \in 3Q$. In either case, we conclude that

$$2^{\nu m} (1 + 2^\nu |x - y|)^{-m} \leq c 2^{-j(m-n)} \chi_{3Q}(x) \frac{\chi_Q(y)}{|Q|}.$$

Next we multiply this inequality by $|g(y)|$ and integrate with respect to y over \mathbb{R}^n . This gives $\eta_{\nu,m} * |g|(x) \leq c 2^{j(m-n)} \chi_{3Q}(x) M_Q g$, which clearly implies the claim. \square

For the proof of the Lemma 5.4 we need the following result on the maximal operator. It follows from Lemma 3.3 and Corollary 3.4, [15], since $p^+ < \infty$ in our case.

Lemma 5.3. *Let $p \in C^{\log}(\mathbb{R}^n)$ with $1 \leq p^- \leq p^+ < \infty$. Then there exists $h \in \text{weak-}L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that*

$$Mf(x)^{p(\cdot)} \leq c M(|f(\cdot)|^{p(\cdot)})(x) + \min\{|Q|, 1\} h(x)$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 1$.

We are now ready for a preliminary version of Theorem 3.2, containing an additional condition.

Lemma 5.4. *Let $p, q \in C^{\log}(\mathbb{R}^n)$, $1 < p^- \leq p^+ < \infty$, $1 < q^- \leq q^+ < \infty$, and $(p/q)^- \cdot q^- > 1$. Then there exists $m > n$ such that*

$$\left\| \left\| \eta_{\nu, m} * f_{\nu} \right\|_{L^{p(\cdot)}_{\nu}} \right\|_{L^{p(\cdot)}_x} \leq c \left\| \left\| f_{\nu} \right\|_{L^{q(\cdot)}_{\nu}} \right\|_{L^{p(\cdot)}_x}$$

for every sequence $\{f_{\nu}\}_{\nu \in \mathbb{N}_0}$ of L^1_{loc} -functions.

Proof. By homogeneity, it suffices to consider the case

$$\left\| \left\| f_{\nu} \right\|_{L^{q(\cdot)}_{\nu}} \right\|_{L^{p(\cdot)}_x} \leq 1.$$

Then, in particular,

$$(5.5) \quad \int_{\mathbb{R}^n} |f_{\nu}(x)|^{p(x)} dx \leq 1$$

for every $\nu \geq 1$. Using Lemma 5.2 and Jensen's inequality (i.e., the embedding in weighted discrete Lebesgue spaces), we estimate

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} |\eta_{\nu, m} * f_{\nu}(x)|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \\ & \leq \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} \left(\sum_{j \geq 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) M_Q f_{\nu} \right)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \\ & \leq c \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} \sum_{j \geq 0} 2^{-j(m-n)} \left(\sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) M_Q f_{\nu} \right)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \\ & \leq c \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} \sum_{j \geq 0} 2^{-j(m-n)} c \sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) (M_Q f_{\nu})^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx. \end{aligned}$$

For the last inequality we used the fact that the innermost sum contains only a finite, uniformly bounded number of non-zero terms.

It follows from (5.5) and $p(x) \geq \frac{q(x)}{q^-}$ that $\|f_{\nu}\|_{L^{\frac{q(\cdot)}{q^-}}} \leq c$. Thus, by Lemma 5.3,

$$(M_Q f_{\nu})^{\frac{q(x)}{q^-}} \leq c M_Q \left(|f_{\nu}|^{\frac{q}{q^-}} \right) + c \min\{|Q|, 1\} h(x)$$

for all $Q \in \mathcal{D}_{\nu-j}$ and $x \in Q$. Combining this with the estimates above, we get

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} |\eta_{\nu} * f_{\nu}(x)|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \\ & \leq c \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} \sum_{j \geq 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) \left[M_Q \left(|f_{\nu}|^{\frac{q}{q^-}} \right) \right]^{q^-} \right)^{\frac{p(x)}{q(x)}} dx \\ & \quad + c \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} \sum_{j \geq 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) (\min\{|Q|, 1\} h(x))^{q^-} \right)^{\frac{p(x)}{q(x)}} dx \\ & =: (I) + (II). \end{aligned}$$

Now we easily estimate that

$$\begin{aligned}
 (I) &\leq c \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} \left[M(|f_\nu|^{\frac{q}{q^-}})(x) \right]^{q^-} \sum_{j \geq 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) \right)^{\frac{p(x)}{q(x)}} dx \\
 &\leq c \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} \left[M(|f_\nu|^{\frac{q}{q^-}})(x) \right]^{q^-} \right)^{\frac{p(x)}{q(x)}} dx \\
 &= c \int_{\mathbb{R}^n} \left\| M(|f_\nu|^{\frac{q}{q^-}})(x) \right\|_{l_{q^-}^{\nu}}^{\frac{p(x)}{q(x)} q^-} dx.
 \end{aligned}$$

The vector valued maximal inequality, Lemma 5.1, with $(p/q)^- \cdot q^- > 1$ and $q^- > 1$, implies that the last expression is bounded since

$$\int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} \left(|f_\nu(x)|^{\frac{q(x)}{q^-}} \right)^{q^-} \right)^{\frac{p(x)}{q(x)}} dx = \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} |f_\nu(x)|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \leq 1.$$

For the estimation of (II) we first note the inequality

$$\begin{aligned}
 \sum_{\nu \geq 0} \sum_{j \geq 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) \min\{|Q|, 1\}^{q^-} &\leq \sum_{\nu \geq 0} \sum_{j \geq 0} 2^{-j(m-n)} \min\{2^{n(j-\nu)q^-}, 1\} \\
 &\leq \sum_{j \geq 0} 2^{-j(m-n)} \left(j + \sum_{\nu > j} 2^{n(j-\nu)q^-} \right) \\
 &\leq \sum_{j \geq 0} 2^{-j(m-n)} (j+1) \leq c.
 \end{aligned}$$

We then estimate (II) as follows:

$$\begin{aligned}
 (II) &\leq c \int_{\mathbb{R}^n} \left(h(x)^{q^-} \sum_{\nu \geq 0} \sum_{j \geq 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) \min\{|Q|^{q^-}, 1\} \right)^{\frac{p(x)}{q(x)}} dx \\
 &\leq c \int_{\mathbb{R}^n} h(x)^{\frac{p(x)}{q(x)} q^-} dx.
 \end{aligned}$$

Since $(p/q)^- \cdot q^- > 1$ and $h \in \text{weak-}L^1 \cap L^\infty$, the last expression is bounded. \square

Using a partitioning trick, it is possible to remove the strange condition $(p/q)^- \cdot q^- > 1$ from the previous lemma and prove our main result regarding multipliers:

Proof of Theorem 3.2. Because of the uniform continuity of p and q , we can choose a finite cover $\{\Omega_i\}$ of \mathbb{R}^n with the following properties:

- (1) each $\Omega_i \subset \mathbb{R}^n$, $1 \leq i \leq k$, is open;
- (2) the sets Ω_i cover \mathbb{R}^n , i.e., $\bigcup_i \Omega_i = \mathbb{R}^n$;
- (3) non-contiguous sets are separated in the sense that $d(\Omega_i, \Omega_j) > 0$ if $|i - j| > 1$; and
- (4) we have $(p/q)_{A_i}^- q_{A_i}^- > 1$ for $1 \leq i \leq k$, where $A_i := \bigcup_{j=i-1}^{i+1} \Omega_j$ (with the understanding that $\Omega_0 = \Omega_{k+1} = \emptyset$).

Let us choose an integer l so that $2^l \leq \min_{|i-j|>1} 3d(\Omega_i, \Omega_j) < 2^{l+1}$. Since there are only finitely many indices, the third condition implies that such an l exists.

Next we split the problem and work with the domains Ω_i . In each of these we argue as in the previous lemma to conclude that

$$\begin{aligned}
 \int_{\mathbb{R}^n} \left(\sum_{\nu \geq 0} |\eta_{\nu,m} * f_\nu(x)|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx &\leq \sum_{i=1}^k \int_{\Omega_i} \left(\sum_{\nu \geq 0} |\eta_{\nu,m} * f_\nu(x)|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \\
 &\leq c \sum_{i=1}^k \int_{\Omega_i} \left(\sum_{\nu \geq 0} \sum_{j \geq 0} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) (M_Q f_\nu)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx.
 \end{aligned}$$

From this we get

$$\begin{aligned} \int_{\Omega_i} \left(\sum_{\nu \geq 0} |\eta_{\nu,m} * f_\nu(x)|^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx &\leq c \int_{\Omega_i} \left(\sum_{\nu \geq 0} \sum_{j=0}^{\nu+l} 2^{-j(m-n)} \sum_{Q \in \mathcal{D}_{\nu-j}} \chi_{3Q}(x) (M_Q f_\nu)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \\ &\quad + c \int_{\Omega_i} \left(\sum_{\nu \geq 0} \sum_{j \geq \nu+l} 2^{-j(m-n)} M f_\nu(x)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx. \end{aligned}$$

The first integral on the right hand side is handled as in the previous proof. This is possible, since the cubes in this integral are always in A_i and $(p/q)_{A_i}^- q_{A_i}^- > 1$.

So it remains only to bound

$$\int_{\Omega_i} \left(\sum_{\nu \geq 0} \sum_{j \geq \nu+l} 2^{-j(m-n)} M f_\nu(x)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \leq c \int_{\Omega_i} \left(\sum_{\nu \geq 0} 2^{-(m-n)\nu} M f_\nu(x)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx.$$

For a non-negative sequence (x_i) we have

$$\left(\sum_{i \geq 0} 2^{-i(m-n)} x_i \right)^r \leq \begin{cases} c(r) \sum_{i \geq 0} 2^{-i(m-n)} x_i^r & \text{if } r \geq 1 \\ \sum_{i \geq 0} 2^{-i(m-n)r} x_i^r, & \text{if } r \leq 1. \end{cases}$$

We apply this estimate for $r = \frac{p(x)}{q(x)}$ and conclude that

$$\int_{\Omega_i} \left(\sum_{\nu \geq 0} 2^{-(m-n)\nu} M f_\nu(x)^{q(x)} \right)^{\frac{p(x)}{q(x)}} dx \leq c \sum_{\nu \geq 0} 2^{-(m-n)\nu \min\{1, (\frac{p}{q})^-\}} \int_{\Omega_i} M f_\nu(x)^{p(x)} dx.$$

The boundedness of the maximal operator implies that the integral may be estimated by a constant, since $\int |f_\nu(x)|^{p(x)} dx \leq 1$. We are left with a geometric sum, which certainly converges. \square

6. PROOFS OF THE DECOMPOSITION RESULTS

We can often take care of the variable smoothness simply by treating it as a constant in a cube, which is what the next lemma is for.

Lemma 6.1. *Let α be as in the Standing Assumptions. There exists $d \in (n, \infty)$ such that if $m > d$, then*

$$2^{\nu\alpha(x)} \eta_{\nu,2m}(x-y) \leq c 2^{\nu\alpha(y)} \eta_{\nu,m}(x-y)$$

for all $x, y \in \mathbb{R}^n$.

Proof. Choose $k \in \mathbb{N}_0$ as small as possible subject to the condition that $|x-y| \leq 2^{-\nu+k}$. Then $1 + 2^\nu |x-y| \approx 2^k$. We estimate that

$$\frac{\eta_{\nu,2m}(x-y)}{\eta_{\nu,m}(x-y)} \leq c (1 + 2^k)^{-m} \leq c 2^{-km}.$$

On the other hand, the log-Hölder continuity of α implies that

$$2^{\nu(\alpha(x)-\alpha(y))} \geq 2^{-\nu c_{\log} / \log(e+1/|x-y|)} \geq 2^{-k c_{\log}} |x-y|^{-c_{\log} / \log(e+1/|x-y|)} \geq c 2^{-k c_{\log}}.$$

The claim follows from these estimates provided we choose $m \geq c_{\log}$. \square

Proof of Theorem 3.4. Let $f \in F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$. Then we have the representation

$$f = \sum_{Q \in \mathcal{D}^+} \langle \varphi_Q, f \rangle \psi_Q = \sum_{Q \in \mathcal{D}^+} |Q|^{\frac{1}{2}} \varphi_\nu * f(x_Q) \psi_Q.$$

Let $r \in (0, \min\{p^-, q^-\})$ and let m be so large that Lemma 6.1 applies. The functions $\varphi_\nu * f$ fulfill the requirements of Lemma A.7, so

$$\begin{aligned} \|S_\varphi f\|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} &= \left\| \left\| 2^{v\alpha(x)} \sum_{Q \in \mathcal{D}_\nu} \varphi_\nu * f(x_Q) \chi_Q \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)}} \\ &\leq c \left\| \left\| 2^{v\alpha(x)} (\eta_{\nu, 2m} * |\varphi_\nu * f|^r)^\frac{1}{r} \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)}} \\ &= c \left\| \left\| 2^{v\alpha(x)r} \eta_{\nu, 2m} * |\varphi_\nu * f|^r \right\|_{l_v^{\frac{q(x)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}}^\frac{1}{r}. \end{aligned}$$

By Lemma 6.1 and Theorem 3.2, we further conclude that

$$\begin{aligned} \|S_\varphi f\|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} &\leq c \left\| \left\| \eta_{\nu, m} * (2^{v\alpha(\cdot)} |\varphi_\nu * f|)^r \right\|_{l_v^{\frac{q(x)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}}^\frac{1}{r} \\ &\leq c \left\| \left\| 2^{v\alpha(x)r} |\varphi_\nu * f|^r \right\|_{l_v^{\frac{q(x)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}}^\frac{1}{r} \\ &= c \left\| \left\| 2^{v\alpha(x)} \varphi_\nu * f \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)}}. \end{aligned}$$

This proves the theorem. \square

In order to prove Theorem 3.8 we need to split our domain into several parts. The following lemma will be applied to each part. For the statement we need Triebel–Lizorkin spaces defined in domains of \mathbb{R}^n . These are achieved simply by replacing $L^{p(\cdot)}(\mathbb{R}^n)$ by $L^{p(\cdot)}(\Omega)$ in the definitions of $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ and $f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$:

$$\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\Omega)} := \left\| \left\| 2^{v\alpha(x)} \varphi_\nu * f(x) \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)}(\Omega)}$$

and

$$\|\{s_Q\}_Q\|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\Omega)} := \left\| \left\| 2^{v\alpha(x)} \sum_{Q \in \mathcal{D}_\nu} |s_Q| |Q|^{-\frac{1}{2}} \chi_Q \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)}(\Omega)}.$$

Lemma 6.2. *Let p , q , and α be as in the Standing Assumptions and define functions $J = n/\min\{1, p, q\}$ and $N = J - n - \alpha$. Let Ω be a cube or the complement of a finite collection of cubes and suppose that $\{m_Q\}_Q$, $Q \subset \Omega$, is a family of $(J^+ - n - \alpha^- + \varepsilon, \alpha^+ + 1 + \varepsilon)$ -smooth molecules, for some $\varepsilon > 0$. Then*

$$\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\Omega)} \leq c \|\{s_Q\}_Q\|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\Omega)}, \quad \text{where } f = \sum_{\nu \geq 0} \sum_{\substack{Q \in \mathcal{D}_\nu \\ Q \subset \Omega}} s_Q m_Q$$

and $c > 0$ is independent of $\{s_Q\}_Q$ and $\{m_Q\}_Q$.

Proof. Let $2m$ be sufficiently large, i.e., larger than M (from the definition of molecules). Choose $r \in (0, \min\{1, p^-, q^-\})$, $\varepsilon > 0$, $k_1 \geq \alpha^+ + 2\varepsilon$ and $k_2 \geq \frac{n}{r} - n - \alpha^- + 2\varepsilon$ so that $\{m_Q\}$ are $(k_2, k_1 + 1, 2m)$ -smooth molecules. Define $k(\nu, \mu) := k_1(\nu - \mu)_+ + k_2(\mu - \nu)_+$ and $\bar{s}_{Q_\mu} := s_{Q_\mu} |Q_\mu|^{-1/2}$.

Next we apply Lemma A.5 twice: with $g = \varphi_\nu$, $h(x) = m_{Q_\mu}(x - x_{Q_\mu})$ and $k = \lfloor k_2 \rfloor + 1$ if $\mu \geq \nu$, and $g(x) = m_{Q_\mu}(x - x_{Q_\mu})$, $h = \varphi_\nu$ and $k = \lfloor k_1 \rfloor + 1$ otherwise. This and Lemma A.2 give

$$\begin{aligned} |\varphi_\nu * m_{Q_\mu}(x)| &\leq c 2^{-k(\nu, \mu)} |Q_\mu|^{1/2} \eta_{\nu, 2m} * \eta_{\mu, 2m}(x + x_{Q_\mu}) \\ &\approx c 2^{-k(\nu, \mu)} |Q_\mu|^{-1/2} (\eta_{\nu, 2m} * \eta_{\mu, 2m} * \chi_{Q_\mu})(x). \end{aligned}$$

Thus, we have

$$\begin{aligned}
\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\Omega)} &= \left\| \left\| \sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} 2^{v\alpha(x)} |s_{Q_\mu}| \varphi_\nu * m_{Q_\mu} \right\|_{l_v^{q(x)}(\Omega)} \right\|_{L_x^{p(\cdot)}(\Omega)} \\
&\leq \left\| \left\| \sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}| 2^{v\alpha(x)-k(v,\mu)} \eta_{v,2m} * \eta_{\mu,2m} * \chi_{Q_\mu} \right\|_{l_v^{q(x)}(\Omega)} \right\|_{L_x^{p(\cdot)}(\Omega)} \\
&= \left\| \left\| \left(\sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}| 2^{v\alpha(x)-k(v,\mu)} \eta_{v,2m} * \eta_{\mu,2m} * \chi_{Q_\mu} \right)^r \right\|_{l_v^{\frac{q(x)}{r}}(\Omega)} \right\|_{L_x^{\frac{p(\cdot)}{r}}(\Omega)}^{\frac{1}{r}}.
\end{aligned}$$

Next we use the embedding $l^r \hookrightarrow l^1$ and obtain the estimate on the term inside of the two norms above as follows

$$\begin{aligned}
&\left(\sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}| 2^{v\alpha(x)-k(v,\mu)} \eta_{v,2m} * \eta_{\mu,2m} * \chi_{Q_\mu} \right)^r \\
&\leq \sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}|^r 2^{v\alpha(x)r-k(v,\mu)r} (\eta_{v,2m} * \eta_{\mu,2m} * \chi_{Q_\mu})^r.
\end{aligned}$$

By Lemma A.4 we conclude that

$$\begin{aligned}
&2^{v\alpha(x)r-k(v,\mu)r} (\eta_{v,2m} * \eta_{\mu,2m} * \chi_{Q_\mu})^r \\
(6.3) \quad &\leq c 2^{v\alpha(x)r-k_1r(v-\mu)_+-k_2r(\mu-\nu)_++n(1-r)(v-\mu)_+} \eta_{v,2mr} * \eta_{\mu,2mr} * \chi_{Q_\mu} \\
&\leq c 2^{\mu\alpha(x)r-2\varepsilon|v-\mu|r} \eta_{v,2mr} * \eta_{\mu,2mr} * \chi_{Q_\mu},
\end{aligned}$$

where, in the second step, we used the assumptions on k_1 and k_2 . We use this with our previous estimate to get

$$\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\Omega)} \leq \left\| \left\| \sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}|^r 2^{\mu\alpha(x)r-2\varepsilon|v-\mu|r} \eta_{v,2mr} * \eta_{\mu,2mr} * \chi_{Q_\mu} \right\|_{l_v^{\frac{q(x)}{r}}(\Omega)} \right\|_{L_x^{\frac{p(\cdot)}{r}}(\Omega)}^{\frac{1}{r}}.$$

We apply Lemma 6.1 and Theorem 3.2 to conclude that

$$\begin{aligned}
\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}(\Omega)} &\leq \left\| \left\| \eta_{v,mr} * \left(\sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}|^r 2^{\mu\alpha(\cdot)r-2\varepsilon|v-\mu|r} \eta_{\mu,2mr} * \chi_{Q_\mu} \right) \right\|_{l_v^{\frac{q(x)}{r}}(\Omega)} \right\|_{L_x^{\frac{p(\cdot)}{r}}(\Omega)}^{\frac{1}{r}} \\
&\leq \left\| \left\| \sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}|^r 2^{\mu\alpha(x)r-2\varepsilon|v-\mu|r} \eta_{\mu,2mr} * \chi_{Q_\mu} \right\|_{l_v^{\frac{q(x)}{r}}(\Omega)} \right\|_{L_x^{\frac{p(\cdot)}{r}}(\Omega)}^{\frac{1}{r}}.
\end{aligned}$$

We estimate the inner part (which depends on x) pointwise as follows:

$$\begin{aligned}
&\left\| \sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}|^r 2^{\mu\alpha(x)r-2\varepsilon|v-\mu|r} \eta_{\mu,2mr} * \chi_{Q_\mu} \right\|_{l_v^{\frac{q(x)}{r}}(\Omega)}^{\frac{q(x)}{r}} \\
&= \sum_{\nu \geq 0} \left| \sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}|^r 2^{\mu\alpha(x)r-2\varepsilon|v-\mu|r} \eta_{\mu,2mr} * \chi_{Q_\mu} \right|_{l_v^{\frac{q(x)}{r}}(\Omega)}^{\frac{q(x)}{r}} \\
&\leq c \sum_{\nu \geq 0} \sum_{\mu \geq 0} 2^{-\varepsilon|v-\mu|r} \left| \sum_{Q_\mu \in \mathcal{D}_\mu} 2^{\mu\alpha(x)r} |\widetilde{s}_{Q_\mu}|^r \eta_{\mu,2mr} * \chi_{Q_\mu} \right|_{l_v^{\frac{q(x)}{r}}(\Omega)}^{\frac{q(x)}{r}},
\end{aligned}$$

where, for the inequality, we used Hölder's inequality in the space with geometrically decaying weight, as in the proof of Theorem 3.2. Now the only part which depends on ν is a geometric sum, which we estimate by a constant. Next we change the power $\alpha(x)$ to $\alpha(y)$

by Lemma 6.1:

$$\begin{aligned}
& \left\| \sum_{\mu \geq 0} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}|^r 2^{\mu\alpha(x)r-2\varepsilon|\nu-\mu|r} \eta_{\mu,2mr} * \chi_{Q_\mu} \right\|_{L_v^{\frac{q(x)}{r}}} \\
& \leq c \sum_{\mu \geq 0} \left\| \sum_{Q_\mu \in \mathcal{D}_\mu} 2^{\mu\alpha(x)r} |\widetilde{s}_{Q_\mu}|^r \eta_{\mu,2mr} * \chi_{Q_\mu} \right\|_{L_v^{\frac{q(x)}{r}}} \\
& \leq c \left\| \eta_{\mu,m} * \left(\sum_{Q_\mu \in \mathcal{D}_\mu} 2^{\mu\alpha(\cdot)r} |\widetilde{s}_{Q_\mu}|^r \chi_{Q_\mu} \right) \right\|_{L_\mu^{\frac{q(x)}{r}}}.
\end{aligned}$$

Hence, we have shown that

$$\|f\|_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\Omega)} \leq c \left\| \eta_{\mu,m} * \left(\sum_{Q \in \mathcal{D}_\mu} 2^{\mu\alpha(\cdot)r} |\widetilde{s}_Q|^r \chi_Q \right) \right\|_{L_\mu^{\frac{q(x)}{r}}}^{\frac{1}{r}} \left\| \right\|_{L_x^{\frac{p(\cdot)}{r}}(\Omega)}.$$

Therefore, by Theorem 3.2, we conclude that

$$\begin{aligned}
\|f\|_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\Omega)} & \leq c \left\| \left\| \sum_{Q \in \mathcal{D}_\mu} 2^{\mu\alpha(x)r} |\widetilde{s}_Q|^r \chi_Q \right\|_{L_\mu^{\frac{q(x)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}(\Omega)}^{\frac{1}{r}} \\
& = c \left\| \left\| \sum_{Q \in \mathcal{D}_\mu} 2^{\mu\alpha(x)r} |s_Q| |Q|^{-\frac{1}{2}} \chi_Q \right\|_{L_\mu^{\frac{q(x)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}(\Omega)} = \| \{s_Q\}_Q \|_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(\Omega)},
\end{aligned}$$

where we used that the sum consists of a single non-zero term. \square

Proof of Theorem 3.8. We will reduce the claim to the previous lemma.

By assumption there exists $\varepsilon > 0$ so that the molecules m_Q are $(N+4\varepsilon, \alpha+1+3\varepsilon)$ -smooth. By the uniform continuity of p, q and α , we may choose $\mu_0 \geq 0$ so that $N_Q^- > J_Q^+ - \alpha_Q^- - n + \varepsilon$ and $\alpha_Q^- > \alpha_Q^+ - \varepsilon$ for every dyadic cube Q of level μ_0 . Note that if Q_0 is a dyadic cube of level μ_0 and $Q \subset Q_0$ is another dyadic cube, then

$$N_Q^- \geq N_{Q_0}^- > J_{Q_0}^+ - \alpha_{Q_0}^- - n - \varepsilon \geq J_Q^+ - \alpha_Q^- - n - \varepsilon,$$

similarly for α . Thus we conclude that m_Q is a $(J_Q^+ - \alpha_Q^- - n + 3\varepsilon, \alpha_Q^+ + 1 + 2\varepsilon)$ -smooth when Q is of level at most μ_0 .

Since p, q and α have a limit at infinity, we conclude that $N_{\mathbb{R}^n \setminus K}^- > J_{\mathbb{R}^n \setminus K}^+ - \alpha_{\mathbb{R}^n \setminus K}^- - n - \varepsilon$ and $\alpha_{\mathbb{R}^n \setminus K}^- > \alpha_{\mathbb{R}^n \setminus K}^+ - \varepsilon$ for some compact set $K \subset \mathbb{R}^n$. We denote by $\Omega_i, i = 1, \dots, M$, those dyadic cubes of level μ_0 which intersect K , and define $\Omega_0 = \mathbb{R}^n \setminus \bigcup_{i=1}^M \Omega_i$.

For every integer $i \in [0, M]$ choose $r_i \in (0, \min\{1, p_{\Omega_i}^-, q_{\Omega_i}^-\})$ so that $\frac{\mu}{r_i} < J_Q^+ + \varepsilon$, and set $k_i := \frac{\mu}{r_i} - n - \alpha_{\Omega_i}^- + 2\varepsilon$ and $K_i = \alpha_{\Omega_i}^+ + 2\varepsilon$. Then m_Q is a $(k_i, K_i + 1)$ -smooth molecule when Q is of level at most μ_0 . Define $k_i(\nu, \mu) := K_i(\nu - \mu)_+ + k_i(\mu - \nu)_+$ and $\widetilde{s}_{Q_\mu} := s_{Q_\mu} |Q_\mu|^{-1/2}$. Finally, let $r \in (0, \min\{1, p^-, q^-\})$.

Note that the constants k_i and K_i have been chosen so that in each set Ω_i we may argue as in the previous lemma. Thus we get

$$|\varphi_\nu * m_{Q_\mu}(x)| \leq c 2^{-k(\nu, \mu)} |Q_\mu|^{-1/2} (\eta_{\nu, 2m} * \eta_{\mu, 2m} * \chi_{Q_\mu})(x).$$

From this we conclude that

$$\begin{aligned}
\|f\|_{F_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}} & \leq \left\| \left\| 2^{\nu\alpha} \varphi_\nu * f \right\|_{L_\nu^{q(x)}} \right\|_{L_x^{p(\cdot)}} \\
& \leq \left\| \left\| \sum_{\mu=0}^{\mu_0-1} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}| 2^{\nu\alpha(x)-k(\nu, \mu)} \eta_{\nu, 2m} * \eta_{\mu, 2m} * \chi_{Q_\mu} \right\|_{L_\nu^{q(x)}} \right\|_{L_x^{p(\cdot)}} \\
& \quad + \sum_{i=0}^M \left\| \left\| \sum_{\mu \geq \mu_0-1} \sum_{Q_\mu \in \mathcal{D}_\mu} |\widetilde{s}_{Q_\mu}| 2^{\nu\alpha(x)-k(\nu, \mu)} \eta_{\nu, 2m} * \eta_{\mu, 2m} * \chi_{Q_\mu} \right\|_{L_\nu^{q(x)}} \right\|_{L_x^{p(\cdot)}(Q_i)}.
\end{aligned}$$

By the previous lemma, each term in the last sum is dominated by $\| \{s_Q\}_Q \|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}$, so we conclude that

$$\begin{aligned} \|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} &\leq \left\| \left\| \sum_{\mu=0}^{\mu_0-1} \sum_{Q_\mu \in \mathcal{D}_\mu} |\bar{s}_{Q_\mu}| 2^{v\alpha(x)-k(v,\mu)} \eta_{v,2m} * \eta_{\mu,2m} * \chi_{Q_\mu} \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)}} \\ &\quad + c(M+1) \| \{s_Q\}_Q \|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}. \end{aligned}$$

It remains only to take care of the first term on the right hand side. An analysis of the proof of the previous lemma shows that the only part where the assumption on the smoothness of the molecules was needed was in the estimate (6.3). In the current case we get instead

$$\begin{aligned} &2^{v\alpha(x)r-rk(v,\mu)} (\eta_{v,2mr} * \eta_{\mu,2mr} * \chi_{Q_\mu})^r \\ &\leq c 2^{\mu\alpha(x)r-2\varepsilon|v-\mu|+n(1-r)_+(\mu-\nu)_+} \eta_{v,2mr} * \eta_{\mu,2mr} * \chi_{Q_\mu}, \end{aligned}$$

since we have no control of k_2 . However, since $\mu \leq \mu_0$ and $\nu \geq 0$, the extra term satisfies $2^{n(1-r)_+(\mu-\nu)_+} \leq 2^{n(1-r)_+\mu_0}$, so it is just a constant. After this modification the rest of the proof of Lemma 6.2 takes care of the first term. \square

Proof of Theorem 3.11. Define constants $K = n / \min\{1, p^-, q^-\} - n + \varepsilon$ and $L = \alpha^+ + 1 + \varepsilon$. We construct (K, L) -smooth atoms $\{a_Q\}_{Q \in \mathcal{D}^+}$ exactly as on p. 132 of [23]. Note that we may use the constant indices construction, since the constants K and L give sufficient smoothness at every point. These atoms are also atoms for the space $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$.

Let $f \in F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$. With functions as in Definition 2.2, we represent f as $f = \sum_{Q \in \mathcal{D}^+} t_Q \varphi_Q$,

where $t_Q = \langle f, \psi_Q \rangle$. Next, we define

$$(t_r^*)_{Q_{vk}} = \left(\sum_{P \in \mathcal{D}_v} \frac{|t_P|^r}{(1 + 2^v |x_P - x_Q|)^m} \right)^{1/r},$$

for $Q = Q_{vk}$, $v \in \mathbb{N}^0$ and $k \in \mathbb{Z}^n$. For these numbers $(t_r^*)_Q$ we know that $f = \sum_Q (t_r^*)_Q a_Q$ where $\{a_Q\}_Q$ are atoms (molecules with support in $3Q$), by the construction of [23]. (Technically, the atoms from the construction of [23] satisfy our inequalities for molecules only up to a constant (independent of the cube and scale). We will ignore this detail.)

For $v \in \mathbb{N}_0$ define $T_v := \sum_{Q \in \mathcal{D}_v} t_Q \chi_Q$. The definition of t_r^* is a discrete convolution of T_v with $\eta_{v,m}$. Changing to the continuous version, we see that $(t_r^*)_{Q_{vk}} \approx (\eta_{v,M} * (|T_v|^r)(x))^{1/r}$ for $x \in Q_{vk}$. By this point-wise estimate we conclude that

$$\begin{aligned} \|t_r^*\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} &= \left\| \left\| \left\{ 2^{v\alpha(x)} \sum_{Q \in \mathcal{D}_v} |Q|^{-\frac{1}{2}} (t_r^*)_Q \chi_Q \right\}_v \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)}} \\ &\approx \left\| \left\| \left\{ 2^{v\alpha(x)+v/2} \eta_{v,M} * (|T_v|^r) \right\}_v \right\|_{l_v^{\frac{q(x)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}}^{\frac{1}{r}}. \end{aligned}$$

Next we use Lemma 6.1 and Theorem 3.2 to conclude that

$$\begin{aligned} &\left\| \left\| \left\{ 2^{v\alpha(x)+v/2} \eta_{v,M} * (|T_v|^r) \right\}_v \right\|_{l_v^{\frac{q(x)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}}^{\frac{1}{r}} \\ &\leq c \left\| \left\| \left\{ 2^{v\alpha(x)+v/2} T_v \right\}_v \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)}} = \left\| \left\| \left\{ 2^{v\alpha(x)} \sum_{Q \in \mathcal{D}_v} |Q|^{-\frac{1}{2}} t_Q \chi_Q \right\}_v \right\|_{l_v^{q(x)}} \right\|_{L_x^{p(\cdot)}}. \end{aligned}$$

Since $f = \sum_{Q \in \mathcal{D}^+} t_Q \varphi_Q$, Theorem 3.4 implies that this is bounded by a constant times $\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}}$.

This completes one direction. The other direction,

$$\|f\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \leq c \| \{s_Q\}_Q \|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}},$$

follows from Theorem 3.8, since every family of atoms is in particular a family of molecules. \square

We next consider a general embedding lemma. The local classical scale of Triebel–Lizorkin spaces is increasing in the primary index p and decreasing in the secondary index q . This is a direct consequence of the corresponding properties of L^p and l^q . In the variable exponent setting we have the following global result provided we assume that p stays constant at infinity:

Proposition 6.4. *Let p_j, q_j , and α_j be as in the Standing Assumptions, $j = 0, 1$.*

- (a) *If $p_0 \geq p_1$ and $(p_0)_\infty = (p_1)_\infty$, then $L^{p_0(\cdot)} \hookrightarrow L^{p_1(\cdot)}$.*
- (b) *If $\alpha_0 \geq \alpha_1$, $p_0 \geq p_1$, $(p_0)_\infty = (p_1)_\infty$, and $q_0 \leq q_1$, then $F_{p_0(\cdot), q_0(\cdot)}^{\alpha_0(\cdot)} \hookrightarrow F_{p_1(\cdot), q_1(\cdot)}^{\alpha_1(\cdot)}$.*

Proof. In Lemma 2.2 of [13] it is shown that $L^{p_0(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_1(\cdot)}(\mathbb{R}^n)$ if and only if $p_0 \geq p_1$ almost everywhere and $1 \in L^{r(\cdot)}(\mathbb{R}^n)$, where $\frac{1}{r(x)} := \frac{1}{p_1(x)} - \frac{1}{p_0(x)}$. Note that $r(x) = \infty$ if $p_1(x) = p_0(x)$. The condition $1 \in L^{r(\cdot)}(\mathbb{R}^n)$ means in this context (since r is usually unbounded) that $\lim_{\lambda \searrow 0} \varrho_{r(\cdot)}(\lambda) = 0$, where we use the convention that $\lambda^{r(x)} = 0$ if $r(x) = \infty$ and $\lambda \in [0, 1)$. Due to the assumptions on p_0 and p_1 , we have $\frac{1}{r} \in C^{\log}$, $\frac{1}{r} \geq 0$, and $\frac{1}{r_\infty} = 0$. In particular, $|\frac{1}{r(x)}| \leq \frac{A}{\log(e+|x|)}$ for some $A > 0$ and all $x \in \mathbb{R}^n$. Thus,

$$\varrho_{r(\cdot)}(\exp(-2nA)) = \int_{\mathbb{R}^n} \exp\left(\frac{-2nA}{|\frac{1}{r(x)}|}\right) dx \leq \int_{\mathbb{R}^n} (e + |x|)^{-2n} dx < \infty.$$

The convexity of $\varrho_{r(\cdot)}$ implies that $\varrho_{r(\cdot)}(\lambda \exp(-2nA)) \rightarrow 0$ as $\lambda \searrow 0$ and (a) follows.

For (b) we argue as follows. Since $\alpha_0 \geq \alpha_1$, we have $2^{v\alpha_0(x)} \leq 2^{v\alpha_1(x)}$ for all $v \geq 0$ and all $x \in \mathbb{R}^n$. Moreover, $q_0 \leq q_1$ implies $\|\cdot\|_{q_1} \leq \|\cdot\|_{q_0}$ and (a) implies $L^{p_0(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_1(\cdot)}(\mathbb{R}^n)$. Now, the claim follows immediately from the definitions of the norms of $F_{p_0(\cdot), q_0(\cdot)}^{\alpha_0(\cdot)}$ and $F_{p_1(\cdot), q_1(\cdot)}^{\alpha_1(\cdot)}$. \square

With the help of this embedding result we can prove the density of smooth functions.

Proof of Corollary 3.12. Choose K so large that $F_{p^+, 2}^K \hookrightarrow F_{p^+, 1}^{\alpha^+}$. This is possible by classical, fixed exponent, embedding results.

Let $f \in F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ and choose smooth atoms $a_Q \in C^k(\mathbb{R}^n)$ so that $f = \sum_{Q \in \mathcal{D}^+} t_Q a_Q$ in \mathcal{S}' . Define

$$f_m = \sum_{v=0}^m \sum_{Q \in \mathcal{D}_v, |x_Q| < m} t_Q a_Q.$$

Then clearly $f_m \in C_0^K$ and $f_m \rightarrow f$ in $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$.

We can choose a sequence of functions $\varphi_{m,k} \in C_0^\infty$ so that $\|f_m - \varphi_{m,k}\|_{W^{K,p^+}} \rightarrow 0$ as $k \rightarrow \infty$ and the support of $\varphi_{m,k}$ is lies in the ball $B(0, r_m)$. By the choice of K we conclude that

$$\|f_m - \varphi_{m,k}\|_{F_{p^+, 1}^{\alpha^+}} \leq c \|f_m - \varphi_{m,k}\|_{F_{p^+, 2}^K} = c \|f_m - \varphi_{m,k}\|_{W^{K,p^+}}.$$

By Proposition 6.4 we conclude that

$$\|f_m - \varphi_{m,k}\|_{F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \leq c \|f_m - \varphi_{m,k}\|_{F_{p^+, 1}^{\alpha^+}}.$$

Note that the assumption $(p_0)_\infty = (p_1)_\infty$ of the proposition is irrelevant, since our functions have bounded support. Combining these inequalities yields that $\varphi_{m,k} \rightarrow f_m$ in $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$, hence we may choose a sequence k_m so that $\varphi_{m,k_m} \rightarrow f$ in $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$, as required. \square

Remark 6.5. Note that we used density of smooth functions in the proof of the equality $F_{p(\cdot), 2}^k \cong W^{k,p(\cdot)}$. However, in the proof of the previous corollary we needed this result only for constant exponent: $F_{p, 2}^k \cong W^{k,p}$. Therefore, the argument is not circular.

7. TRACES

In this section we deal with trace theorems for Triebel–Lizorkin spaces. We write \mathcal{D}^n and \mathcal{D}_v^n for the families of dyadic cubes in \mathcal{D}^+ when we want to emphasize the dimension of the underlying space. The idea of the proof of the main trace theorem is to use the localization afforded by the atomic decomposition, and express a function as a sum of only those atoms with support intersecting the hyperplane $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. In the classical case, this approach is due to Frazier and Jawerth [22].

There have been other approaches to deal with traces and extension operators using wavelet decomposition instead of atomic decomposition, which utilizes compactly supported Daubechies wavelets, and thus, conveniently gives trace theorems (see, e.g., [25]). However, for that one would need to define and establish properties of almost diagonal operators and almost diagonal matrices for the $F_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ and $f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ spaces. In the interest of brevity we leave this for future research.

The following lemma shows that it does not matter much for the norm if we shift around the mass a bit in the sequence space.

Lemma 7.1. *Let p, q , and α be as in the Standing Assumptions, $\varepsilon > 0$, and let $\{E_Q\}_Q$ be a collection of sets with $E_Q \subset 3Q$ and $|E_Q| \geq \varepsilon |Q|$. Then*

$$\| \{s_Q\}_Q \|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} \approx \left\| \left\| 2^{v\alpha(x)} \sum_{Q \in \mathcal{D}_v} |s_Q| |Q|^{-\frac{1}{2}} \chi_{E_Q} \right\|_{l_v^{q(\cdot)}} \right\|_{L_x^{p(\cdot)}}$$

for all $\{s_Q\}_Q \in f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$.

Proof. We start by proving the inequality “ \leq ”. Let $r \in (0, \min\{p^-, q^-\})$. We express the norm as

$$\| \{s_Q\}_Q \|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \left\| \left\| 2^{v\alpha(x)r} \sum_{Q \in \mathcal{D}_v} |s_Q|^r |Q|^{-\frac{r}{2}} \chi_Q \right\|_{l_v^{\frac{q(\cdot)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}},$$

since the sum has only one non-zero term. We use the estimate $\chi_Q \leq c \eta_{v,m} * \chi_{E_Q}$ for all $Q \in \mathcal{D}_v$. Now Lemma 6.1 implies that

$$\begin{aligned} \| \{s_Q\}_Q \|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} &\leq c \left\| \left\| 2^{v\alpha(x)r} \sum_{Q \in \mathcal{D}_v} |s_Q|^r |Q|^{-\frac{r}{2}} \eta_v * \chi_{E_Q} \right\|_{l_v^{\frac{q(\cdot)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}} \\ &\leq c \left\| \left\| \eta_v * \left(2^{v\alpha(\cdot)r} \sum_{Q \in \mathcal{D}_v} |s_Q|^r |Q|^{-\frac{r}{2}} \chi_{E_Q} \right) \right\|_{l_v^{\frac{q(\cdot)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}}. \end{aligned}$$

Then Theorem 3.2 completes the proof of the first direction:

$$\begin{aligned} \| \{s_Q\}_Q \|_{f_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} &\leq c \left\| \left\| 2^{v\alpha(x)r} \sum_{Q \in \mathcal{D}_v} |s_Q|^r |Q|^{-\frac{r}{2}} \chi_{E_Q} \right\|_{l_v^{\frac{q(\cdot)}{r}}} \right\|_{L_x^{\frac{p(\cdot)}{r}}} \\ &= c \left\| \left\| 2^{v\alpha(x)} \sum_{Q \in \mathcal{D}_v} |s_Q| |Q|^{-\frac{1}{2}} \chi_{E_Q} \right\|_{l_v^{q(\cdot)}} \right\|_{L_x^{p(\cdot)}}. \end{aligned}$$

The other direction follows by the same argument, since $\chi_{E_Q} \leq c \eta_v * \chi_Q$. \square

Next we use the embedding proposition from the previous section to show that the trace space does not really depend on the secondary index of integration.

Lemma 7.2. *Let p_1, p_2, q_1, α_1 and α_2 be as in the Standing Assumptions and let $q_2 \in (0, \infty)$. Assume that $\alpha_1 = \alpha_2$ and $p_1 = p_2$ in the upper or lower half space, and that $\alpha_1 \geq \alpha_2$ and $p_1 \leq p_2$. Then*

$$\mathrm{tr} F_{p_1(\cdot), q_1(\cdot)}^{\alpha_1(\cdot)}(\mathbb{R}^n) = \mathrm{tr} F_{p_2(\cdot), q_2}^{\alpha_2(\cdot)}(\mathbb{R}^n).$$

Proof. We assume without loss of generality that $\alpha_1 = \alpha_2$ and $p_1 = p_2$ in the upper half space. We define $r_0 = \min\{q_2, q_1^-\}$ and $r_1 = \max\{q_2, q_1^+\}$. It follows from Proposition 6.4 that

$$\mathrm{tr} F_{p_2(\cdot), r_0}^{\alpha_2(\cdot)} \hookrightarrow \mathrm{tr} F_{p_1(\cdot), q_1(\cdot)}^{\alpha_2(\cdot)} \hookrightarrow \mathrm{tr} F_{p_1(\cdot), r_1}^{\alpha_1(\cdot)}$$

and

$$\mathrm{tr} F_{p_2(\cdot), r_0}^{\alpha_2(\cdot)} \hookrightarrow \mathrm{tr} F_{p_2(\cdot), q_2}^{\alpha_2(\cdot)} \hookrightarrow \mathrm{tr} F_{p_1(\cdot), r_1}^{\alpha_1(\cdot)}.$$

We complete the proof by showing that $\mathrm{tr} F_{p_1(\cdot), r_1}^{\alpha_1(\cdot)} \hookrightarrow \mathrm{tr} F_{p_2(\cdot), r_0}^{\alpha_2(\cdot)}$. Let $f \in \mathrm{tr} F_{p_1(\cdot), r_1}^{\alpha_1(\cdot)}$. According to Theorem 3.11 we have the representation

$$f = \sum_{Q \in \mathcal{D}^+} t_Q a_Q \quad \text{with} \quad \|\{t_Q\}_Q\|_{f_{p_1(\cdot), r_1}^{\alpha_1(\cdot)}} \leq c \|f\|_{F_{p_1(\cdot), r_1}^{\alpha_1(\cdot)}},$$

where the a_Q are smooth atoms for $F_{p_1(\cdot), r_1}^{\alpha_1(\cdot)}$ satisfying (M1) and (M2) up to high order. Then they are also smooth atoms for $F_{p_2(\cdot), r_0}^{\alpha_2(\cdot)}$.

Let $A := \{Q \in \mathcal{D}^+ : 3\overline{Q} \cap \{x_n = 0\} \neq \emptyset\}$. If $Q \in A$ is contained in the closed upper half space, then we write $Q \in A^+$, otherwise $Q \in A^-$. We set $\tilde{t}_Q = t_Q$ when $Q \in A$, and $\tilde{t}_Q = 0$ otherwise. Then we define $\tilde{f} = \sum_{Q \in \mathcal{D}^+} \tilde{t}_Q \tilde{a}_Q$. It is clear that $\mathrm{tr} f = \mathrm{tr} \tilde{f}$, since all the atoms of f whose support intersects \mathbb{R}^{n-1} are included in \tilde{f} . For $Q \in A^+$ we define

$$E_Q = \{x \in Q : \tfrac{3}{4}\ell(Q) \leq x_n \leq \ell(Q)\};$$

for $Q \in A^-$ we define

$$E_Q = \{(x', x_n) \in \mathbb{R}^n : (x', -x_n) \in Q, \tfrac{1}{2}\ell(Q) \leq x_n \leq \tfrac{3}{4}\ell(Q)\};$$

for all other cubes $E_Q = \emptyset$. If $Q \in A$, then $|Q| = 4|E_Q|$; moreover, $\{E_Q\}_Q$ covers each point at most three times.

By Theorem 3.8 and Lemma 7.1 we conclude that

$$\|\tilde{f}\|_{F_{p_2(\cdot), r_0}^{\alpha_2(\cdot)}} \leq c \|\{\tilde{t}_Q\}_Q\|_{f_{p_2(\cdot), r_0}^{\alpha_2(\cdot)}} \leq c \left\| \left\| 2^{v_{\alpha_2}(x)} \sum_{Q \in \mathcal{D}_v} |t_Q| |Q|^{-\frac{1}{2}} \chi_{E_Q} \right\|_{l_v^1} \right\|_{L_x^{p_2(\cdot)}}.$$

The inner norm consists of at most three non-zero members for each $x \in \mathbb{R}^n$. Therefore, we can replace r_0 by r_1 . Moreover, each E_Q is supported in the upper half space, where α_2 and α_1 , and p_2 and p_1 agree. Thus,

$$\|\tilde{f}\|_{F_{p_2(\cdot), r_0}^{\alpha_2(\cdot)}} \leq c \left\| \left\| 2^{v_{\alpha_1}(x)} \sum_{Q \in \mathcal{D}_v} |t_Q| |Q|^{-\frac{1}{2}} \chi_{E_Q} \right\|_{l_v^1} \right\|_{L_x^{p_1(\cdot)}}.$$

The right hand side is bounded by $\|f\|_{F_{p_1(\cdot), r_1}^{\alpha_1(\cdot)}}$ according to Theorem 3.8 and Lemma 7.1.

Therefore, $\mathrm{tr} F_{p_1(\cdot), r_1}^{\alpha_1(\cdot)} \hookrightarrow \mathrm{tr} F_{p_2(\cdot), r_0}^{\alpha_2(\cdot)}$, and the claim follows. \square

For the next proposition we recall the common notation $F_{p(\cdot)}^{\alpha(\cdot)} = F_{p(\cdot), p(\cdot)}^{\alpha(\cdot)}$ for the Triebel–Lizorkin space with identical primary and secondary indices of integrability. The next result shows that the trace space depends only on the values of the indices at the boundary, as should be expected.

Proposition 7.3. *Let p_1, p_2, q_1, α_1 and α_2 be as in the Standing Assumptions. Assume that $\alpha_1(x) = \alpha_2(x)$ and $p_1(x) = p_2(x)$ for all $x \in \mathbb{R}^{n-1} \times \{0\}$. Then*

$$\mathrm{tr} F_{p_1(\cdot), q_1(\cdot)}^{\alpha_1(\cdot)}(\mathbb{R}^n) = \mathrm{tr} F_{p_2(\cdot)}^{\alpha_2(\cdot)}(\mathbb{R}^n).$$

Proof. By Lemma 7.2 we conclude that $\mathrm{tr} F_{p_1(\cdot), q_1(\cdot)}^{\alpha_1(\cdot)} = \mathrm{tr} F_{p_1(\cdot)}^{\alpha_1(\cdot)}$. Therefore, we can assume that $q_1 = p_1$.

We define $\widetilde{\alpha}_j$ to equal α_j on the lower half space and $\min\{\alpha_1, \alpha_2\}$ on the upper half space and let $\widetilde{\alpha} = \min\{\alpha_1, \alpha_2\}$. Similarly, we define \widetilde{p}_j and \widetilde{p} . Applying Lemma 7.2 four times in the following chain

$$\mathrm{tr} F_{p_1(\cdot)}^{\alpha_1(\cdot)}(\mathbb{R}^n) = \mathrm{tr} F_{\widetilde{p}_1(\cdot)}^{\widetilde{\alpha}_1(\cdot)}(\mathbb{R}^n) = \mathrm{tr} F_{\widetilde{p}(\cdot)}^{\widetilde{\alpha}(\cdot)}(\mathbb{R}^n) = \mathrm{tr} F_{\widetilde{p}_2(\cdot)}^{\widetilde{\alpha}_2(\cdot)}(\mathbb{R}^n) = \mathrm{tr} F_{p_2(\cdot)}^{\alpha_2(\cdot)}(\mathbb{R}^n),$$

gives the result. \square

Proof of Theorem 3.13. By Proposition 7.3 it suffices to consider the case $q = p$ with p and α independent of the n -th coordinate for $|x_n| \leq 2$. Let $f \in F_{p(\cdot)}^{\alpha(\cdot)}$ with $\|f\|_{F_{p(\cdot)}^{\alpha(\cdot)}} \leq 1$ and let $f = \sum s_Q a_Q$ be an atomic decomposition as in Theorem 3.11.

We denote by π the orthogonal projection of \mathbb{R}^n onto \mathbb{R}^{n-1} , and $(x', x_n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. For $J \in \mathcal{D}_\mu^{n-1}$, a dyadic cube in \mathbb{R}^{n-1} , we define $Q_i(J) \in \mathcal{D}_\mu^n$, $i = 1, \dots, 6 \cdot 5^{n-1}$, to be all the dyadic cubes satisfying $J \subset 3Q_i$. We define $t_J = |Q_1(J)|^{-\frac{1}{2n}} \sum_i |s_{Q_i(J)}|$ and $h_J(x') = t_J^{-1} \sum_i s_{Q_i} a_{Q_i}$. By $Q_+(J)$ we denote the cube $Q_i(J)$ which has J as a face (i.e. $J \subset \partial Q_+(J)$). Then we have

$$\mathrm{tr} f(x') = \sum_\mu \sum_{J \in \mathcal{D}_\mu^{n-1}} t_J h_J(x'),$$

with convergence in \mathcal{S}' . The condition $\alpha - \frac{1}{p} - (n-1)\left(\frac{1}{p} - 1\right)_+ > 0$ implies that molecules in $F_{p(\cdot)}^{\alpha(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1})$ are not required to satisfy any moment conditions. Therefore, h_J is a family of smooth molecules for this space. Consequently, by Theorem 3.8, we find that

$$\|\mathrm{tr} f\|_{F_{p(\cdot)}^{\alpha(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1})} \leq c \| \{t_J\}_J \|_{f_{p(\cdot)}^{\alpha(\cdot) - \frac{1}{p(\cdot)}}(\mathbb{R}^{n-1})}.$$

Thus, we conclude the proof by showing that the right hand side is bounded by a constant. Since the norm is bounded if and only if the modular is bounded, we see that it suffices to show that

$$\begin{aligned} & \int_{\mathbb{R}^{n-1}} \sum_\mu \sum_{J \in \mathcal{D}_\mu^{n-1}} \left(2^{\mu(\alpha(x',0) - \frac{1}{p(x',0)})} |t_J| |J|^{-1/2} \chi_J(x', 0) \right)^{p(x',0)} dx' \\ &= \sum_\mu \sum_{J \in \mathcal{D}_\mu^{n-1}} 2^{-\mu} \int_J \left(2^{\mu\alpha(x',0)} |t_J| |J|^{-1/2} \right)^{p(x',0)} dx' \end{aligned}$$

is bounded. For the integral we calculate

$$\begin{aligned} 2^{-\mu} \int_J \left(2^{\mu\alpha(x',0)} |t_J| |J|^{-1/2} \right)^{p(x',0)} dx' &= \int_{Q_+(J)} \left(2^{\mu\alpha(x',0)} |t_J| |J|^{-1/2} \right)^{p(x',0)} d(x', x_n) \\ &\leq c \int_{Q_+(J)} \left(2^{\mu\alpha(x)} \sum_i |s_{Q_i}| |Q|^{-\frac{1}{2n} - \frac{n-1}{2n}} \right)^{p(x)} dx \\ &= c \int_{Q_+(J)} \left(2^{\mu\alpha(x)} \sum_i |s_{Q_i}| |Q|^{-1/2} \right)^{p(x)} dx \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \sum_\mu \sum_{J \in \mathcal{D}_\mu^{n-1}} 2^{-\mu} \int_J \left(2^{\mu\alpha(x',0)} |t_J| |J|^{-1/2} \right)^{p(x',0)} dx' \\ &\leq c \sum_\mu \sum_{Q \in \mathcal{D}_\mu^n} \int_Q \left(2^{\mu\alpha(x)} \sum_i |s_{Q_i}| |Q|^{-1/2} \right)^{p(x)} dx \\ &\leq c \int_{\mathbb{R}^n} \sum_v \sum_{Q \in \mathcal{D}_v^n} \left(2^{v\alpha(x)} |s_Q| |Q|^{-1/2} \chi_Q(x) \right)^{p(x)} dx, \end{aligned}$$

where we again swapped the integral and the sums. Since $\|f\|_{F_{p(\cdot)}^{\alpha(\cdot)}} \leq 1$, the right hand side quantity is bounded, and we are done. \square

APPENDIX A. TECHNICAL LEMMAS

Recall from (3.1) that $\eta_{v,m}(x) = 2^{nv}(1 + 2^v|x|)^{-m}$.

Lemma A.1. *Let $v_1 \geq v_0$, $m > n$, and $y \in \mathbb{R}^n$. Then*

$$\begin{aligned} \eta_{v_0,m}(y) &\leq 2^m \eta_{v_1,m}(y) && \text{if } |y| \leq 2^{-v_1}; \text{ and} \\ \eta_{v_1,m}(y) &\leq 2^m \eta_{v_0,m}(y) && \text{if } |y| \geq 2^{-v_0}. \end{aligned}$$

Proof. Let $|y| \leq 2^{-v_1}$. Then $1 + 2^{v_1}|y| \leq 2$ and

$$\frac{\eta_{v_0,m}(y)}{\eta_{v_1,m}(y)} = \frac{2^{nv_0}(1 + 2^{v_1}|y|)^m}{2^{nv_1}(1 + 2^{v_0}|y|)^m} \leq \frac{2^{nv_0} \cdot 2^m}{2^{nv_1}} \leq 2^m,$$

which proves the first inequality. Assume now that $|y| \geq 2^{-v_0}$. Then $1 + 2^{v_0}|y| \leq 2 \cdot 2^{v_0}|y|$ and

$$\frac{\eta_{v_1,m}(y)}{\eta_{v_0,m}(y)} = \frac{2^{nv_1}(1 + 2^{v_0}|y|)^m}{2^{nv_0}(1 + 2^{v_1}|y|)^m} \leq \frac{2^{nv_1}(2 \cdot 2^{v_0}|y|)^m}{2^{nv_0}(2^{v_1}|y|)^m} = 2^m 2^{(v_1-v_0)(n-m)} \leq 2^m,$$

which gives the second inequality. \square

Lemma A.2. *Let $v \geq 0$ and $m > n$. Then for $Q \in \mathcal{D}_v$, $y \in Q$ and $x \in \mathbb{R}^n$, we have*

$$\eta_{v,m} * \left(\frac{\chi_Q}{|Q|} \right)(x) \approx \eta_{v,m}(x - y).$$

Proof. Fix $Q \in \mathcal{D}_v$ and set $d = 1 + \sqrt{n}$. If $y, z \in Q$, then $|y - z| \leq \sqrt{n} 2^{-v}$ and

$$\begin{aligned} \frac{1}{d} (1 + 2^v|x - z|) &\leq 1 + \frac{1}{d} \cdot 2^v(|x - z| - \sqrt{n} 2^{-v}) \\ &\leq 1 + 2^v|x - y| \\ &\leq 1 + 2^v(|x - z| + \sqrt{n} 2^{-v}) \leq d(1 + 2^v|x - z|). \end{aligned}$$

Therefore, for all $y, z \in Q$ we have

$$2^{-m} \eta_{v,m}(x - y) \leq \eta_{v,m}(x - z) \leq 2^m \eta_{v,m}(x - y).$$

The claim follows when we integrate this estimate over $z \in Q$ and use the formula

$$\eta_{v,m} * \left(\frac{\chi_Q}{|Q|} \right)(x) = \frac{1}{|Q|} \int_Q \eta_{v,m}(x - z) dz. \quad \square$$

Lemma A.3. *For $v_0, v_1 \geq 0$ and $m > n$, we have*

$$\eta_{v_0,m} * \eta_{v_1,m} \approx \eta_{\min\{v_0, v_1\}, m}$$

with the constant depending only on m and n .

Proof. Using dilations and symmetry we may assume that $v_0 = 0$ and $v_1 \geq 0$. Since $m > n$, we have $\|\eta_{v_0,m}\|_1 \leq c$ and $\|\eta_{v_1,m}\|_1 \leq c$.

We start with the direction “ \geq ”. If $|y| \leq 2^{-v_1} \leq 1$, then $1 + |x - y| \leq 2(1 + |x|)$, and therefore, $\eta_{v_0,m}(x - y) \geq c \eta_{v_0,m}(x)$. Hence,

$$\begin{aligned} \eta_{v_0,m} * \eta_{v_1,m}(x) &\geq \int_{\{y: |y| \leq 2^{-v_1}\}} \eta_{v_0,m}(x - y) \eta_{v_1,m}(y) dy \\ &\geq c \eta_{v_0,m}(x) \int_{\{y: |y| \leq 2^{-v_1}\}} 2^{nv_1}(1 + 2^{v_1}|y|)^{-m} dy \\ &\geq c \eta_{v_0,m}(x) \int_{\{y: |y| \leq 2^{-v_1}\}} 2^{nv_1} 2^{-m} dy \\ &\geq c 2^{-m} \eta_{v_0,m}(x). \end{aligned}$$

We now prove the opposite direction, “ \leq ”. Let $A := \{y \in \mathbb{R}^n : |y| \leq 3 \text{ or } |x - y| > |x|/2\}$. If $y \in A$, then $1 + |x - y| \geq \frac{1}{4}(1 + |x|)$, which implies that $\eta_{v_0,m}(x - y) \leq c \eta_{v_0,m}(x)$ and

$$\int_A \eta_{v_0,m}(x - y) \eta_{v_1,m}(y) dy \leq c \eta_{v_0,m}(x) \int_A \eta_{v_1,m}(y) dy \leq c \eta_{v_0,m}(x).$$

If $y \in \mathbb{R}^n \setminus A$, then $|y| \geq 1$ and $|y| \geq \frac{1}{2}|x|$. So $\eta_{v_1,m}(y) \leq c \eta_{v_0,m}(y) \leq c \eta_{v_0,m}(x)$ by Lemma A.1. Hence,

$$\int_{\mathbb{R}^n \setminus A} \eta_{v_0,m}(x - y) \eta_{v_1,m}(y) dy \leq c \int_{\mathbb{R}^n \setminus A} \eta_{v_0,m}(x - y) dy \eta_{v_0,m}(x) \leq c \eta_{v_0,m}(x).$$

Combining the estimates over A and $\mathbb{R}^n \setminus A$ gives

$$\eta_{v_0,m} * \eta_{v_1,m}(x) \leq c \eta_{\min\{v_0, v_1\}, m}(x). \quad \square$$

Lemma A.4. *Let $r \in (0, 1]$. Then for $v, \mu \geq 0$, $m > \frac{n}{r}$ and $Q_\mu \in \mathcal{D}_\mu$, we have*

$$(\eta_{v,m} * \eta_{\mu,m} * \chi_{Q_\mu})^r \approx 2^{(\mu-v)+n(1-r)} \eta_{v,mr} * \eta_{\mu,mr} * \chi_{Q_\mu},$$

where the constant depends only on m, n and r .

Proof. Without loss of generality, we may assume that $x_{Q_\mu} = 0$. Then by Lemmas A.2 and A.3

$$\begin{aligned} \eta_{v,m} * \eta_{\mu,m} * \chi_{Q_\mu} &\approx 2^{-n\mu} \eta_{v,m} * \eta_{\mu,m} \approx 2^{-n\mu} \eta_{\min\{v,\mu\},m}, \\ \eta_{v,mr} * \eta_{\mu,mr} * \chi_{Q_\mu} &\approx 2^{-n\mu} \eta_{v,mr} * \eta_{\mu,mr} \approx 2^{-n\mu} \eta_{\min\{v,\mu\},mr}. \end{aligned}$$

From the definition of η we get

$$(\eta_{\min\{v,\mu\},m})^r = 2^{\min\{v,\mu\}n(r-1)} \eta_{\min\{v,\mu\},mr}.$$

Thus, we get

$$(\eta_{v,m} * \eta_{\mu,m} * \chi_{Q_\mu})^r \approx 2^{\mu n(1-r)} 2^{\min\{v,\mu\}n(r-1)} \eta_{v,mr} * \eta_{\mu,mr} * \chi_{Q_\mu}. \quad \square$$

Lemma A.5. *Let $g, h \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $k \in \mathbb{N}_0$ such that $D^\mu g \in L^1(\mathbb{R}^n)$ for all multi-indices μ with $|\mu| \leq k$. Assume that there exist $m_0 > n$ and $m_1 > n + k$ such that $|h| \leq \eta_{\mu,m_1}$ and $|D^\mu g| \leq 2^{\nu k} \eta_{v,m_0}$. Further, suppose that*

$$\int_{\mathbb{R}^n} x^\gamma h(x) dx = 0, \quad \text{for } |\gamma| \leq k - 1.$$

Then $|g * h| \leq c 2^{k(v-\mu)} \eta_{v,m_0} * \eta_{\mu,m_1-k}$.

Proof. If $k = 0$, then the estimate is obvious, so we can assume $k \geq 1$. It suffices to prove the result for g, h smooth. Since h has vanishing moments up to order $k - 1$ we estimate by Taylor's formula

$$\begin{aligned} |g * h(x)| &\leq \int_{\mathbb{R}^n} \left| \left(g(y) - \sum_{|\gamma| \leq k-1} D^\gamma g(x) \frac{(y-x)^\gamma}{\gamma!} \right) h(x-y) \right| dy \\ &\leq c \int_{\mathbb{R}^n} \int_{[x,y]} \sup_{|\mu|=k} |D^\mu g(\xi)| |x - \xi|^{k-1} d\xi |h(x-y)| dy \\ &\leq c \int_{\mathbb{R}^n} \int_{[x,y]} 2^{\nu k} \eta_{v,m_0}(\xi) |x - \xi|^{k-1} \eta_{\mu,m_1}(x-y) d\xi dy \end{aligned}$$

Changing the order of integration with $y - x = r(\xi - x)$, where $r \geq 1$, yields the inequality

$$(A.6) \quad |g * h(x)| \leq c 2^{\nu k} \int_{\mathbb{R}^n} \int_1^\infty \eta_{v,m_0}(\xi) |x - \xi|^k \eta_{\mu,m_1}(r(x - \xi)) dr d\xi.$$

We estimate the inner integral: for $2^\mu r|x - \xi| \geq 1$ we have $\eta_{\mu, m_1}(r(x - \xi)) \approx r^{-m_1} \eta_{\mu, m_1}(x - \xi)$; for $2^\mu r|x - \xi| < 1$ we simply use $\eta_{\mu, m_1}(r(x - \xi)) \leq \eta_{\mu, m_1}(x - \xi)$. Thus, we find that

$$\begin{aligned} \int_1^\infty \eta_{\mu, m_1}(r(x - \xi)) dr &\leq \left(\int_1^\infty r^{-1-m_1} dr + \int_1^{2^{-\mu}|x-\xi|^{-1}} r^{-1} dr \right) \eta_{\mu, m_1}(x - \xi) \\ &\approx \log(e + 2^{-\mu}|x - \xi|^{-1}) \eta_{\mu, m_1}(x - \xi). \end{aligned}$$

Substituting this into (A.6) produces

$$\begin{aligned} |g * h(x)| &\leq c 2^{\nu k} \int_{\mathbb{R}^n} \eta_{\nu, m_0}(\xi) |x - \xi|^k \log(e + 2^{-\mu}|x - \xi|^{-1}) \eta_{\mu, m_1}(x - \xi) d\xi \\ &= c 2^{\nu k} \int_{\mathbb{R}^n} \log(e + 2^{-\mu}|x - \xi|^{-1}) \left(\frac{|x - \xi|}{1 + 2^\mu|x - \xi|} \right)^k \eta_{\nu, m_0}(\xi) \eta_{\mu, m_1-k}(x - \xi) d\xi \\ &\leq 2^{k(\nu-\mu)} \eta_{\nu, m_0} * \eta_{\mu, m_1-k}(x), \end{aligned}$$

proving the assertion. \square

Lemma A.7 ("The r -trick"). *Let $r > 0$, $\nu \geq 0$ and $m > n$. Let $x \in \mathbb{R}^n$. Then there exists $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'$ with $\text{supp } \hat{g} \subset \{\xi : |\xi| \leq 2^{\nu+1}\}$, we have*

$$|g(x)| \leq c (\eta_{\nu, m} * |g|^r(x))^{1/r}.$$

Proof. Fix a dyadic cube $Q = Q_{\nu, k}$ and $x \in Q$. By (2.11) of [22] we have

$$g(x) \leq \sup_{z \in Q} |g(z)|^r \leq c_r 2^{\nu n} \sum_{l \in \mathbb{Z}^n} (1 + |l|)^{-m} \int_{Q_{\nu, k+l}} |g(y)|^r dy.$$

In the reference this was shown only for $m = n + 1$, but it is easy to see that it is also true for $m > n + 1$. Now for $x \in Q_{\nu, k}$ and $y \in Q_{\nu, k+l}$, we have $|x - y| \approx 2^{-\nu}|l|$ for large l , hence, $1 + 2^\nu|x - y| \approx 1 + |l|$. From this we conclude that

$$\begin{aligned} \sup_{z \in Q} |g(z)|^r &\leq c_{r, n} \sum_{l \in \mathbb{Z}^n} \int_{Q_{\nu, k+l}} (1 + 2^\nu|x - y|)^{-m} |g(y)|^r dy \\ &= c_{r, n} \int_{\mathbb{R}^n} 2^{\nu n} (1 + 2^\nu|x - z|)^{-m} |g(z)|^r dz = c_{r, n} \eta_{\nu, m} * |g|^r(x). \end{aligned}$$

Now, taking the r -th root, we obtain the claim. \square

Acknowledgment. We would like to thank H.-G. Leopold for useful discussions on how our spaces relate to his spaces of variable smoothness, and Tuomas Hytönen for a piece of advice on Fourier analysis.

The first author thanks Arizona State University and the University of Oulu for their hospitality. All authors thank the W. Pauli Institute, Vienna, at which it was possible to complete the project.

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